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Non-perturbative Heat Kernel Asymptotics on Homogeneous Abelian Bundles

Ivan G. Avramidi and Guglielmo Fucci

*New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA*

E-mail: iavramid@nmt.edu, gfucci@nmt.edu

We study the heat kernel for a Laplace type partial differential operator acting on smooth sections of a complex vector bundle with the structure group $G \times U(1)$ over a Riemannian manifold M without boundary. The total connection on the vector bundle naturally splits into a G -connection and a $U(1)$ -connection, which is assumed to have a parallel curvature F . We find a new local short time asymptotic expansion of the off-diagonal heat kernel $U(t|x, x')$ close to the diagonal of $M \times M$ assuming the curvature F to be of order t^{-1} . The coefficients of this expansion are polynomial functions in the Riemann curvature tensor (and the curvature of the G -connection) and its derivatives with universal coefficients depending in a non-polynomial but analytic way on the curvature F , more precisely, on tF . These functions generate all terms quadratic and linear in the Riemann curvature and of arbitrary order in F in the usual heat kernel coefficients. In that sense, we effectively sum up the usual short time heat kernel asymptotic expansion to all orders of the curvature F . We compute the first three coefficients (both diagonal and off-diagonal) of this new asymptotic expansion.

1 Introduction

The heat kernel is one of the most powerful tools in quantum field theory and quantum gravity as well as mathematical physics and differential geometry (see for example [18, 23, 11, 12, 10, 20, 22, 19] and further references therein). It is of particular importance because the heat kernel methods give a framework for manifestly covariant calculation of a wide range of relevant quantities in quantum field theory like one-loop effective action, Green's functions, effective potential etc.

Unfortunately the exact computation of the heat kernel can be carried out only for exceptional highly symmetric cases when the spectrum of the operator is known exactly, (see [17, 19, 20] and the references in [8, 15, 14, 13]). Although these special cases are very important, in quantum field theory we need the effective action, and, therefore, the heat kernel for general background fields. For this reason various approximation schemes have been developed. One of the oldest methods is the Minackshisundaram-Pleijel short-time asymptotic expansion of the heat kernel as $t \rightarrow 0$ (see the references in [18, 2, 23]).

Despite its enormous importance, this method is essentially perturbative. It is an expansion in powers of the curvatures R and their derivatives and, hence, is inadequate for large curvatures when $tR \sim 1$. To be able to describe the situation when at least some of the curvatures are large one needs an essentially *non-perturbative approach*, which effectively sums up in the short time asymptotic expansion of the heat kernel an infinite series of terms of certain structure that contain large curvatures (for a detailed analysis see [4, 9] and reviews [10, 12]). For example, the partial summation of higher derivatives enables one to obtain a non-local expansion of the heat kernel in powers of curvatures (high-energy approximation in physical terminology). This is still an essentially perturbative approach since the curvatures (but not their derivatives) are assumed to be small and one expands in powers of curvatures.

On another hand to study the situation when curvatures (but not their derivatives) are large (low energy approximation) one needs an essentially *non-perturbative approach*. A promising approach to the calculation of the low-energy heat kernel expansion was developed in non-Abelian gauge theories and quantum gravity in [3, 4, 5, 6, 7, 8, 13, 14, 15]. While the papers [3, 4, 6, 7] dealt with the parallel $U(1)$ -curvature (that is, constant electromagnetic field) in flat space, the papers [5, 8, 13] dealt with symmetric spaces (pure gravitational field in absence of an electromagnetic field). The difficulty of combining the gauge fields and gravity was finally overcome in the papers [14, 15], where homogeneous bundles with

parallel curvature on symmetric spaces was studied.

In this paper we compute the heat kernel for the covariant Laplacian with a large parallel $U(1)$ curvature F in a Riemannian manifold (that is, strong covariantly constant electromagnetic field in an arbitrary gravitational field). Our aim is to evaluate the first three coefficients of the heat kernel asymptotic expansion in powers of Riemann curvature R but *in all orders* of the $U(1)$ curvature F . This is equivalent to a partial summation in the heat kernel asymptotic expansion as $t \rightarrow 0$ of all powers of F in terms which are linear and quadratic in Riemann curvature R .

2 Setup of the Problem

Let M be a n -dimensional compact Riemannian manifold without boundary and \mathcal{S} be a complex vector bundle over M realizing a representation of the group $G \otimes U(1)$. Let φ be a section of the bundle \mathcal{S} and ∇ be the total connection on the bundle \mathcal{S} (including the G -connection as well as the $U(1)$ -connection). Then the commutator of covariant derivatives defines the curvatures

$$[\nabla_\mu, \nabla_\nu]\varphi = (\mathcal{R}_{\mu\nu} + iF_{\mu\nu})\varphi, \quad (2.1)$$

where $\mathcal{R}_{\mu\nu}$ is the curvature of the G -connection and $F_{\mu\nu}$ is the curvature of the $U(1)$ -connection (which will be also called the electromagnetic field).

In the present paper we consider a second-order Laplace type partial differential operator,

$$\mathcal{L} = -\Delta, \quad \Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu. \quad (2.2)$$

The heat kernel for the operator \mathcal{L} is defined as the solution of the heat equation

$$(\partial_t + \mathcal{L})U(t|x, x') = 0, \quad (2.3)$$

with the initial condition

$$U(0|x, x') = \mathcal{P}(x, x')\delta(x, x'). \quad (2.4)$$

where $\delta(x, x')$ is the covariant scalar delta function and $\mathcal{P}(x, x')$ is the operator of parallel transport of the sections of the bundle \mathcal{S} along the geodesic from the point x' to the point x .

The spectral properties of the operator \mathcal{L} are described in terms of the spectral functions, defined in terms of the L^2 traces of some functions of the operator \mathcal{L} ,

such as the zeta-function $\zeta(s) = \text{Tr } \mathcal{L}^{-s}$, and the heat trace

$$\text{Tr } \exp(-t\mathcal{L}) = \int_M d\text{vol } \text{tr } U^{\text{diag}}(t), \quad (2.5)$$

where $d\text{vol} = g^{1/2}dx$ is the Riemannian volume element with $g = \det g_{\mu\nu}$ and tr denotes the fiber trace. Here and everywhere below the diagonal value of any two point quantity $f(x, x')$ denotes the coincidence limit as $x \rightarrow x'$, that is,

$$f^{\text{diag}} = f(x, x). \quad (2.6)$$

It is well known [18] that the heat kernel has the asymptotic expansion as $t \rightarrow 0$ (see also [2, 10, 11, 23])

$$U(t|x, x') \sim (4\pi t)^{-n/2} \mathcal{P}(x, x') \Delta^{1/2}(x, x') \exp \left[-\frac{\sigma(x, x')}{2t} \right] \sum_{k=0}^{\infty} t^k a_k(x, x'), \quad (2.7)$$

where $\sigma(x, x')$ is the geodesic interval (or the world function) defined as one half the square of the geodesic distance between the points x and x' and $\Delta(x, x')$ is the Van Vleck-Morette determinant. The coefficients $a_k(x, x')$ are called the off-diagonal heat kernel coefficients.

The heat kernel diagonal and the heat trace have the asymptotic expansion as $t \rightarrow 0$ [2, 23]

$$U^{\text{diag}}(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k a_k^{\text{diag}}, \quad (2.8)$$

$$\text{Tr } \exp(-t\mathcal{L}) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k A_k, \quad (2.9)$$

where

$$a_k^{\text{diag}} = a_k(x, x) \quad (2.10)$$

and

$$A_k = \int_M d\text{vol } \text{tr } a_k^{\text{diag}}. \quad (2.11)$$

The coefficients A_k are called the global heat kernel coefficients; they are spectral invariants of the operator \mathcal{L} .

The diagonal heat kernel coefficients a_k^{diag} are polynomials in the jets of the metric, the G -connection and the $U(1)$ -connection; in other words, in the curvature tensors and their derivatives. Let us symbolically denote the jets of the metric and the G -connection by

$$R_{(n)} = \left\{ \nabla_{(\mu_1} \cdots \nabla_{\mu_n} R^a_{\mu_{n+1} \mu_{n+2}} \right\}, \quad (2.12)$$

and the jets of the $U(1)$ connection by

$$F_{(n)} = \nabla_{(\mu_1} \cdots \nabla_{\mu_n} F^a_{\mu_{n+1}}). \quad (2.13)$$

Here and everywhere below the parenthesis indicate complete symmetrization over all indices included.

By counting the dimension it is easy to describe the general structure of the coefficients a_k^{diag} . Let us introduce the multi-indices of nonnegative integers

$$\mathbf{i} = (i_1, \dots, i_m), \quad \mathbf{j} = (j_1, \dots, j_l). \quad (2.14)$$

Let us also denote

$$|\mathbf{i}| = i_1 + \cdots + i_m, \quad |\mathbf{j}| = j_1 + \cdots + j_l. \quad (2.15)$$

Then symbolically

$$a_k^{\text{diag}} = \sum_{N=1}^k \sum_{l=0}^N \sum_{m=0}^{N-l} \sum_{\substack{\mathbf{i}, \mathbf{j} \geq 0 \\ |\mathbf{i}| + |\mathbf{j}| + 2N = 2k}} C_{(k,l,m), \mathbf{i}, \mathbf{j}} F_{(j_1)} \cdots F_{(j_l)} R_{(i_1)} \cdots R_{(i_m)}, \quad (2.16)$$

where $C_{(k,l,m), \mathbf{i}, \mathbf{j}}$ are some universal constants.

The lower order diagonal heat kernel coefficients are well known [18, 2, 11]

$$a_0^{\text{diag}} = 1, \quad (2.17)$$

$$a_1^{\text{diag}} = \frac{1}{6}R, \quad (2.18)$$

$$\begin{aligned} a_2^{\text{diag}} = & \frac{1}{30}\Delta R + \frac{1}{72}R^2 - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \\ & + \frac{1}{12}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{1}{6}\mathcal{R}_{\mu\nu}iF^{\mu\nu} - \frac{1}{12}F_{\mu\nu}F^{\mu\nu}. \end{aligned} \quad (2.19)$$

To avoid confusion we should stress that the normalization of the coefficients a_k differs from the papers [2, 10, 11].

In the present paper we study the case of a *parallel* $U(1)$ curvature (covariantly constant electromagnetic field), i.e.

$$\nabla_\mu F_{\alpha\beta} = 0 . \quad (2.20)$$

That is, all jets $F_{(n)}$ are set to zero except the one of order zero, which is F itself. In this case eq. (2.16) takes the form

$$a_k^{\text{diag}} = \sum_{N=1}^k \sum_{l=0}^N \sum_{m=0}^{N-l} \sum_{\substack{\mathbf{i} \geq 0 \\ |\mathbf{i}| + 2N = 2k}} C_{(k,l,m),\mathbf{i}} F^l R_{(i_1)} \cdots R_{(i_m)} , \quad (2.21)$$

where $C_{(k,l,m),\mathbf{i}}$ are now some (other) numerical coefficients.

Thus, by summing up all powers of F in the asymptotic expansion of the heat kernel diagonal we obtain a *new (non-perturbative) asymptotic expansion*

$$U^{\text{diag}}(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \tilde{a}_k^{\text{diag}}(t) , \quad (2.22)$$

where the coefficients $\tilde{a}_k^{\text{diag}}(t)$ are polynomials in the jets $R_{(n)}$

$$\tilde{a}_k^{\text{diag}}(t) = \sum_{N=1}^k \sum_{m=0}^N \sum_{\substack{\mathbf{i} \geq 0 \\ |\mathbf{i}| + 2N = 2k}} f_{(m,\mathbf{i})}^{(k)}(t) R_{(i_1)} \cdots R_{(i_m)} , \quad (2.23)$$

and $f_{(m,\mathbf{i})}^{(k)}(t)$ are some universal dimensionless tensor-valued analytic functions that depend on F only in the dimensionless combination tF .

For the heat trace we obtain then a new asymptotic expansion of the form

$$\text{Tr} \exp(-t\mathcal{L}) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \tilde{A}_k(t) , \quad (2.24)$$

where

$$\tilde{A}_k(t) = \int_M d\text{vol} \, \text{tr} \tilde{a}_k^{\text{diag}}(t) . \quad (2.25)$$

This expansion can be described more rigorously as follows. We rescale the $U(1)$ -curvature F by

$$F \mapsto F(t) = t^{-1} \tilde{F} , \quad (2.26)$$

so that $tF(t) = \tilde{F}$ is independent of t . Then the operator $\mathcal{L}(t)$ becomes dependent on t (in a singular way!). However, the heat trace still has a nice asymptotic expansion as $t \rightarrow 0$

$$\text{Tr} \exp[-t\mathcal{L}(t)] \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \tilde{A}_k, \quad (2.27)$$

where the coefficients \tilde{A}_k are expressed in terms of $\tilde{F} = tF(t)$, and, therefore, are independent of t . Thus, what we are doing is the *asymptotic expansion of the heat trace for a particular case of a singular (as $t \rightarrow 0$) time-dependent operator $\mathcal{L}(t)$* .

Let us stress once again that the eq. (2.23) should not be taken literally; it only represents the general structure of the coefficients $\tilde{a}_k^{\text{diag}}(t)$. To avoid confusion we list below the general structure of the low-order coefficients in more detail

$$\tilde{a}_0^{\text{diag}}(t) = f^{(0)}(t), \quad (2.28)$$

$$\tilde{a}_1^{\text{diag}}(t) = f_{(1,1)}^{(1) \alpha\beta\mu\nu}(t) R_{\alpha\beta\mu\nu} + f_{(1,2)}^{(1) \mu\nu}(t) \mathcal{R}_{\mu\nu}, \quad (2.29)$$

$$\begin{aligned} \tilde{a}_2^{\text{diag}}(t) = & f_{(1,1)}^{(2) \alpha\beta\mu\nu\sigma\rho}(t) \nabla_{(\alpha} \nabla_{\beta)} R_{\mu\nu\sigma\rho} + f_{(1,2)}^{(2) \alpha\beta\mu\nu}(t) \nabla_{(\alpha} \nabla_{\beta)} \mathcal{R}_{\mu\nu} \\ & + f_{(2,1)}^{(2) \alpha\beta\gamma\delta\mu\nu\sigma\rho}(t) R_{\alpha\beta\gamma\delta} R_{\mu\nu\sigma\rho} + f_{(2,2)}^{(2) \alpha\beta\mu\nu}(t) \mathcal{R}_{\alpha\beta} \mathcal{R}_{\mu\nu} \\ & + f_{(2,3)}^{(2) \alpha\beta\mu\nu\sigma\rho}(t) \mathcal{R}_{\alpha\beta} R_{\mu\nu\sigma\rho} \end{aligned} \quad (2.30)$$

with obvious enumeration of the functions. It is the *universal tensor functions* $f_{(l,m)}^{(i)}(t)$ that are of prime interest in this paper. Our main goal is to compute the functions $f_{(l,m)}^{(i)}(t)$ for the coefficients $\tilde{a}_0^{\text{diag}}(t)$, $\tilde{a}_1^{\text{diag}}(t)$ and $\tilde{a}_2^{\text{diag}}(t)$.

Of course, for $t = 0$ (or $F = 0$) the coefficients $\tilde{a}_k(t)$ are equal to the usual diagonal heat kernel coefficients

$$\tilde{a}_k^{\text{diag}}(0) = a_k^{\text{diag}}. \quad (2.31)$$

Therefore, by using the explicit form of the coefficients a_k^{diag} given by (2.19) we obtain the initial values for the functions $f_{(j,k)}^{(i)}$. Moreover, by analyzing the corresponding terms in the coefficients a_3^{diag} and a_4^{diag} (which are known, [18, 2, 22]), one can obtain partial information about some lower order Taylor coefficients of

the functions $f_{(j,k)}^{(i)}(t)$:

$$f^{(0)}(t) = 1 - \frac{1}{12}t^2 F_{\mu\nu} F^{\mu\nu} + O(t^3), \quad (2.32)$$

$$f_{(1,1)}^{(1)\alpha\beta}{}_{\mu\nu}(t) = \frac{1}{6}\delta_{[\mu}^{\alpha}\delta_{\nu]}^{\beta} + O(t), \quad (2.33)$$

$$f_{(1,2)}^{(1)\mu\nu}(t) = \frac{1}{6}tiF^{\mu\nu} + O(t^2), \quad (2.34)$$

$$f_{(1,1)}^{(2)\alpha\beta\mu\nu}{}_{\sigma\rho}(t) = \frac{1}{30}g^{\alpha\beta}\delta_{[\sigma}^{\mu}\delta_{\rho]}^{\nu} + O(t), \quad (2.35)$$

$$f_{(1,2)}^{(2)\alpha\beta}{}_{\mu\nu}(t) = -\frac{1}{15}tiF^{(\alpha}{}_{[\nu}\delta_{\mu]}^{\beta)} + O(t^2), \quad (2.36)$$

$$\begin{aligned} f_{(2,1)\alpha\beta}^{(2)\gamma\delta}{}_{\mu\nu}{}^{\sigma\rho}(t) &= \frac{1}{180}g_{\mu[\alpha}g_{\beta]\nu}g^{\sigma[\gamma}g^{\delta]\rho} - \frac{1}{180}\delta_{[\alpha}^{[\gamma}\delta_{\beta][\nu}g^{\delta][\rho}\delta_{\mu]}^{\sigma]} \\ &\quad + \frac{1}{72}\delta_{[\alpha}^{\gamma}\delta_{\beta]}^{\delta}\delta_{[\mu}^{\sigma}\delta_{\nu]}^{\rho} + O(t), \end{aligned} \quad (2.37)$$

$$f_{(2,2)}^{(2)\alpha\beta}{}_{\mu\nu}(t) = \frac{1}{12}\delta_{[\mu}^{\alpha}\delta_{\nu]}^{\beta} + O(t), \quad (2.38)$$

$$\begin{aligned} f_{(2,3)}^{(2)\alpha\beta\mu\nu}{}_{\sigma\rho}(0) &= -\frac{1}{36}tiF^{\alpha\beta}\delta_{[\sigma}^{\mu}\delta_{\rho]}^{\nu} - \frac{1}{30}tiF^{\mu\nu}\delta_{[\sigma}^{\alpha}\delta_{\rho]}^{\beta} + \frac{1}{9}\delta_{[\sigma}^{[\mu}tiF^{\nu][\alpha}\delta_{\rho]}^{\beta]} + O(t^2). \end{aligned} \quad (2.39)$$

This information can be used to check our final results.

Notice that the global coefficients $\tilde{A}_k(t)$ have exactly the same form as the local ones; the only difference is that the terms with the derivatives of the Riemann curvature do not contribute to the integrated coefficients since they can be eliminated by integrating by parts and taking into account that F is covariantly constant.

Moreover, we study even more general non-perturbative asymptotic expansion for the *off-diagonal* heat kernel and compute the coefficients of zero, first and second order in the Riemann curvature. We will show that there is a *new non-perturbative asymptotic expansion* of the off-diagonal heat kernel as $t \rightarrow 0$ (and

$F = t^{-1}\tilde{F}$, so that tF is fixed) of the form

$$U(t|x, x') \sim \mathcal{P}(x, x')\Delta^{1/2}(x, x')U_0(t|x, x') \sum_{k=0}^{\infty} t^{k/2} b_k(t|x, x') \quad (2.40)$$

where U_0 is an analytic function of F such that for $F = 0$

$$U_0(t|x, x') \Big|_{F=0} = (4\pi t)^{-n/2} \exp \left[-\frac{\sigma(x, x')}{2t} \right]. \quad (2.41)$$

Here $b_k(t|x, x')$ are analytic functions of t that depend on F only in the dimensionless combination tF . Of course, for $t = 0$ they are equal to the usual heat kernel coefficients, that is,

$$b_{2k}(0|x, x') = a_k(x, x'), \quad b_{2k+1}(0|x, x') = 0. \quad (2.42)$$

Moreover, we will show below that the odd-order coefficients vanish not only for $t = 0$ and any $x \neq x'$ but also for any t and $x = x'$, that is, on the diagonal,

$$b_{2k+1}^{\text{diag}}(t) = 0. \quad (2.43)$$

Thus, the heat kernel diagonal has the asymptotic expansion (2.22) as $t \rightarrow 0$ with

$$\tilde{a}_k^{\text{diag}}(t) = (4\pi t)^{n/2} U_0^{\text{diag}}(t) b_{2k}^{\text{diag}}(t). \quad (2.44)$$

3 Geometric Framework

Our goal is to study the heat kernel $U(t|x, x')$ in the neighborhood of the diagonal as $x \rightarrow x'$. Therefore, we will expand all relevant quantities in covariant Taylor series near the diagonal following the methods developed in [2, 11, 10, 12]. We fix a point, say x' , on the manifold M and consider a sufficiently small neighborhood of x' , say a geodesic ball with a radius smaller than the injectivity radius of the manifold. Then, there exists a unique geodesic that connects every point x to the point x' . In order to avoid a cumbersome notation, we will denote by *Latin letters* tensor indices associated to the point x and by *Greek letters* tensor indices associated to the point x' . Of course, the indices associated with the point x (resp. x') are raised and lowered with the metric at x (resp. x'). Also, we will denote by ∇_a (resp. ∇'_μ) covariant derivative with respect to x (resp. x').

We remind below the definition of some of the two-point functions that we will need in our analysis. First of all, the world function $\sigma(x, x')$ is defined as one half of the square of the length of the geodesic between the points x and x' . It satisfies the equation

$$\sigma = \frac{1}{2} u^a u_a = u_\mu u^\mu, \quad (3.1)$$

where

$$u_a = \nabla_a \sigma, \quad u_\mu = \nabla'_\mu \sigma. \quad (3.2)$$

The variables u^μ are nothing but the normal coordinates at the point x' .

The Van Vleck-Morette determinant is defined by

$$\Delta(x, x') = g^{-\frac{1}{2}}(x) \det[-\nabla_a \nabla'_\nu \sigma(x, x')] g^{-\frac{1}{2}}(x'). \quad (3.3)$$

This quantity should not be confused with the Laplacian $\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu$. Usually, the meaning of Δ will be clear from the context. We find it convenient to parameterize it by

$$\Delta(x, x') = \exp[2\zeta(x, x')]. \quad (3.4)$$

Next, we define the tensor

$$\eta^\mu{}_b = \nabla_b \nabla'^\mu \sigma, \quad (3.5)$$

and the tensor $\gamma^a{}_\mu$ inverse to $\eta^\mu{}_a$ by

$$\gamma^a{}_\mu \eta^\mu{}_b = \delta^a_b, \quad \eta^\mu{}_b \gamma^b{}_\nu = \delta^\mu_\nu. \quad (3.6)$$

This enables us to define new derivative operators by

$$\bar{\nabla}_\mu = \gamma^a{}_\mu \nabla_a. \quad (3.7)$$

These operators commute when acting on objects that have been parallel transported to the point x' (in other words the objects that do not have Latin indices). In fact, when acting on such objects these operators are just partial derivatives with respect to normal coordinate u

$$\bar{\nabla}_\mu = \frac{\partial}{\partial u^\mu}. \quad (3.8)$$

We also define the operators

$$\mathcal{D}_\mu = \bar{\nabla}_\mu - \frac{1}{2} i F_{\mu\alpha} u^\alpha. \quad (3.9)$$

Obviously, they form the algebra

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = iF_{\mu\nu}, \quad [\mathcal{D}_\mu, u^\nu] = \delta_\mu^\nu. \quad (3.10)$$

Next, the parallel displacement operator $\mathcal{P}(x, x')$ of sections of the vector bundle \mathcal{S} along the geodesic from the point x' to the point x is defined as the solution of the equation

$$u^a \nabla_a \mathcal{P}(x, x') = 0, \quad (3.11)$$

with the initial condition

$$\mathcal{P}(x, x) = \mathbb{I}, \quad (3.12)$$

where \mathbb{I} is the identity endomorphism of the bundle \mathcal{S} . Finally, we define the two-point quantity

$$\mathcal{A}_\mu = \mathcal{P}^{-1} \bar{\nabla}_\mu \mathcal{P}. \quad (3.13)$$

We remind, here, that we consider the case of a covariantly constant electromagnetic field, i.e.

$$\nabla_\mu F_{\alpha\beta} = 0. \quad (3.14)$$

In this case we find it useful to decompose the quantity \mathcal{A}_μ as

$$\mathcal{A}_\mu = -\frac{1}{2} i F_{\mu\alpha} u^\alpha + \tilde{\mathcal{A}}_\mu. \quad (3.15)$$

By using this machinery we can rewrite the heat kernel as follows. First of all, the heat kernel can be presented in the form

$$U(t|x, x') = \exp(-t\mathcal{L}) \mathcal{P}(x, x') \delta(x, x'), \quad (3.16)$$

which can also be written as

$$U(t|x, x') = \mathcal{P}(x, x') \Delta^{\frac{1}{2}}(x, x') \exp(-t\tilde{\mathcal{L}}) \delta(u), \quad (3.17)$$

where $\delta(u)$ is the usual delta-function in the normal coordinates u^μ (recall that u^μ depends on x and x' and $u = 0$ when $x = x'$) and $\tilde{\mathcal{L}}$ is an operator defined by

$$\tilde{\mathcal{L}} = \mathcal{P}^{-1}(x, x') \Delta^{-\frac{1}{2}}(x, x') \mathcal{L} \Delta^{\frac{1}{2}}(x, x') \mathcal{P}(x, x'). \quad (3.18)$$

As is shown in [2, 11] the operator $\tilde{\mathcal{L}}$ can be written in the form

$$\begin{aligned} \tilde{\mathcal{L}} &= -\Delta^{\frac{1}{2}} (\bar{\nabla}_\mu + \mathcal{A}_\mu) \Delta^{-1} X^{\mu\nu} (\bar{\nabla}_\nu + \mathcal{A}_\nu) \Delta^{\frac{1}{2}} \\ &= -(\bar{\nabla}_\mu + \mathcal{A}_\mu - \zeta_\mu) X^{\mu\nu} (\bar{\nabla}_\nu + \mathcal{A}_\nu + \zeta_\nu) \\ &= -(\mathcal{D}_\mu + \tilde{\mathcal{A}}_\mu - \zeta_\mu) X^{\mu\nu} (\mathcal{D}_\nu + \tilde{\mathcal{A}}_\nu + \zeta_\nu), \end{aligned} \quad (3.19)$$

where $\zeta_\mu = \bar{\nabla}_\mu \zeta$.

Now, by using these equations and by recalling the formula in (3.15), one can rewrite the operator in (3.19) in another way as follows

$$\tilde{\mathcal{L}} = - \left(X^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu + Y^\mu \mathcal{D}_\mu + Z \right), \quad (3.20)$$

where

$$X^{\mu\nu} = \eta^\mu_a \eta^{\nu a}, \quad (3.21)$$

$$Y^\mu = (\bar{\nabla}_\mu X^{\mu\nu}) + 2X^{\mu\nu} \bar{\mathcal{A}}_\mu, \quad (3.22)$$

$$\begin{aligned} Z = & \bar{\mathcal{A}}_\mu X^{\mu\nu} \bar{\mathcal{A}}_\nu - \zeta_\mu X^{\mu\nu} \zeta_\nu + (\bar{\nabla}_\mu X^{\mu\nu}) \bar{\mathcal{A}}_\nu + (\bar{\nabla}_\mu X^{\mu\nu}) \zeta_\nu \\ & + X^{\mu\nu} \bar{\nabla}_\mu \bar{\mathcal{A}}_\nu + X^{\mu\nu} \bar{\nabla}_\mu \zeta_\nu. \end{aligned} \quad (3.23)$$

4 Perturbation Theory

Our goal is now to develop the perturbation theory for the heat kernel. We need to identify a small expansion parameter ε in which the perturbation theory will be organized as $\varepsilon \rightarrow 0$. First of all, we assume that t is small, more precisely, we require $t \sim \varepsilon^2$. Also, since we will work close to the diagonal, that is, x is close to x' , we require that $u^\mu \sim \varepsilon$. This will also mean that $\bar{\nabla} \sim \varepsilon^{-1}$ and $\partial_t \sim \varepsilon^{-2}$. Finally, we assume that F is large, that is, of order $F \sim \varepsilon^{-2}$. To summarize,

$$t \sim \varepsilon^2, \quad u^\mu \sim \varepsilon, \quad F \sim \varepsilon^{-2}. \quad (4.1)$$

4.1 Covariant Taylor Expansion

The Taylor expansions of the quantities introduced above have the form (up to the fifth order) [2, 11]

$$\begin{aligned} X^{\mu\nu} = & g^{\mu\nu} + \frac{1}{3} R^\mu_{\alpha}{}^{\nu}{}_{\beta} u^\alpha u^\beta - \frac{1}{6} \nabla_\alpha R^\mu_{\beta}{}^{\nu}{}_{\gamma} u^\alpha u^\beta u^\gamma + \frac{1}{20} \nabla_\alpha \nabla_\beta R^\mu_{\gamma}{}^{\nu}{}_{\delta} u^\alpha u^\beta u^\gamma u^\delta \\ & + \frac{1}{15} R^\mu_{\alpha\lambda\beta} R^\lambda_{\gamma}{}^{\nu}{}_{\delta} u^\alpha u^\beta u^\gamma u^\delta + O(u^5), \end{aligned} \quad (4.2)$$

$$\begin{aligned} \zeta = & \frac{1}{12} R_{\alpha\beta} u^\alpha u^\beta - \frac{1}{24} \nabla_\alpha R_{\beta\gamma} u^\alpha u^\beta u^\gamma + \frac{1}{80} \nabla_\alpha \nabla_\beta R_{\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta \\ & + \frac{1}{360} R_{\mu\alpha\nu\beta} R^\mu_{\gamma}{}^{\nu}{}_{\delta} u^\alpha u^\beta u^\gamma u^\delta + O(u^5), \end{aligned} \quad (4.3)$$

$$\begin{aligned}\Delta^{\frac{1}{2}} &= 1 + \frac{1}{12}R_{\alpha\beta}u^\alpha u^\beta - \frac{1}{24}\nabla_\alpha R_{\beta\gamma}u^\alpha u^\beta u^\gamma + \frac{1}{80}\nabla_\alpha \nabla_\beta R_{\gamma\delta}u^\alpha u^\beta u^\gamma u^\delta \\ &+ \frac{1}{288}R_{\alpha\beta}R_{\gamma\delta}u^\alpha u^\beta u^\gamma u^\delta + \frac{1}{360}R_{\mu\alpha\nu\beta}R^\mu{}^\nu{}_\gamma{}_\delta u^\alpha u^\beta u^\gamma u^\delta + O(u^5) .\end{aligned}\quad (4.4)$$

$$\begin{aligned}\tilde{\mathcal{A}}_\mu &= -\frac{1}{2}\mathcal{R}_{\mu\alpha}u^\alpha + \frac{1}{24}R_{\mu\alpha\nu\beta}iF^\nu{}_\gamma u^\alpha u^\beta u^\gamma + \frac{1}{3}\nabla_\alpha \mathcal{R}_{\mu\beta}u^\alpha u^\beta \\ &+ \frac{1}{24}R_{\mu\alpha\nu\beta}\mathcal{R}^\nu{}_\gamma u^\alpha u^\beta u^\gamma - \frac{1}{8}\nabla_\alpha \nabla_\beta \mathcal{R}_{\mu\gamma}u^\alpha u^\beta u^\gamma \\ &- \frac{1}{720}R_{\mu\alpha\nu\beta}R^\nu{}_{\gamma\lambda\delta}iF^\lambda{}_\epsilon u^\alpha u^\beta u^\gamma u^\delta u^\epsilon + O(u^6) .\end{aligned}\quad (4.5)$$

We would like to stress that all coefficients of such expansions are evaluated at the point x' . Also note that, the expansion for $\tilde{\mathcal{A}}_\mu$ is valid in the case of a covariantly constant electromagnetic field.

4.2 Perturbation Theory for the Operator \mathcal{L}

Now, we expand the operator $\tilde{\mathcal{L}}$ in a formal power series in ε (recall that $\mathcal{D} \sim \varepsilon^{-1}$ and $u \sim \varepsilon$) to obtain

$$\mathcal{L} \sim - \sum_{k=0}^{\infty} \mathcal{L}_k , \quad (4.6)$$

where \mathcal{L}_k are operators of order ε^{k-2} . In particular,

$$\mathcal{L}_0 = \mathcal{D}^2 , \quad (4.7)$$

$$\mathcal{L}_1 = 0 , \quad (4.8)$$

$$\mathcal{L}_k = X_k^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu + Y_k^\mu \mathcal{D}_\mu + Z_k , \quad k \geq 2 . \quad (4.9)$$

where

$$\mathcal{D}^2 = g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu , \quad (4.10)$$

and $X_k^{\mu\nu}$, Y_k^μ and Z_k are some tensor-valued polynomials in normal coordinates u^μ .

Note that $X_k^{\mu\nu}$ are homogeneous polynomials in normal coordinates u^μ and F of order ε^k . Similarly, $Y_k^\mu \sim \varepsilon^{k-1}$ and $Z_k \sim \varepsilon^{k-2}$. Of course, here the terms Fuu are counted as of order zero. That is, they have the form

$$X_k^{\mu\nu} = P_{(1),k}^{\mu\nu}(u) , \quad (4.11)$$

$$Y_k^\mu = P_{(2),k-1}^\mu + F_{\alpha\beta} P_{(3),k+1}^{\mu\alpha\beta}(u) , \quad (4.12)$$

$$Z_k = P_{(4),k-2} + F_{\alpha\beta} P_{(5),k}^{\alpha\beta}(u) + F_{\alpha\beta} F_{\rho\sigma} P_{(6),k+2}^{\alpha\beta\rho\sigma}(u) , \quad (4.13)$$

where $P_{(j),k}(u)$ are homogeneous tensor valued polynomials of degree k .

By using the covariant Taylor expansions in (4.2), (4.5) and (4.3) we find the explicit expression of the coefficients

$$X_2^{\mu\nu} = C_2^{\mu\nu}{}_{\alpha\beta} u^\alpha u^\beta, \quad (4.14)$$

$$Y_2^\mu = E_2^\mu{}_\alpha u^\alpha + G_2^\mu{}_{\alpha\beta\gamma} u^\alpha u^\beta u^\gamma, \quad (4.15)$$

$$Z_2 = H_2{}_{\alpha\beta} u^\alpha u^\beta + L_2, \quad (4.16)$$

$$X_3^{\mu\nu} = C_3^{\mu\nu}{}_{\alpha\beta\gamma} u^\alpha u^\beta u^\gamma, \quad (4.17)$$

$$Y_3^\mu = E_3^\mu{}_{\alpha\beta} u^\alpha u^\beta, \quad (4.18)$$

$$Z_3 = H_3{}_\alpha u^\alpha, \quad (4.19)$$

$$X_4^{\mu\nu} = C_4^{\mu\nu}{}_{\alpha\beta\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta, \quad (4.20)$$

$$Y_4^\mu = E_4^\mu{}_{\alpha\beta\gamma} u^\alpha u^\beta u^\gamma + G_4^\mu{}_{\alpha\beta\gamma\delta\epsilon} u^\alpha u^\beta u^\gamma u^\delta u^\epsilon, \quad (4.21)$$

$$Z_4 = H_4{}_{\alpha\beta} u^\alpha u^\beta + L_4{}_{\alpha\beta\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta + O_4{}_{\alpha\beta\gamma\delta\epsilon\kappa} u^\alpha u^\beta u^\gamma u^\delta u^\epsilon u^\kappa, \quad (4.22)$$

where

$$\begin{aligned} C_2^{\mu\nu}{}_{\alpha\beta} &= \frac{1}{3} R^\mu{}_{(\alpha}{}^\nu{}_{\beta)}, \\ E_2^\mu{}_\alpha &= -\frac{1}{3} R^\mu{}_\alpha - \mathcal{R}^\mu{}_\alpha, \\ G_2^\mu{}_{\alpha\beta\gamma} &= -\frac{1}{12} R^\mu{}_{(\alpha}{}^\nu{}_{\beta} iF_{\gamma)\nu}, \\ H_2{}_{\alpha\beta} &= -\frac{1}{24} R_{\mu(\alpha} iF^\mu{}_{\beta)}, \\ L_2 &= \frac{1}{6} R, \end{aligned} \quad (4.23)$$

$$\begin{aligned} C_3^{\mu\nu}{}_{\alpha\beta\gamma} &= -\frac{1}{6} \nabla_{(\alpha} R^\mu{}_{\beta}{}^\nu{}_{\gamma)}, \\ E_3^\mu{}_{\alpha\beta} &= \frac{1}{3} \nabla_{(\alpha} R^\mu{}_{\beta)} - \frac{1}{6} \nabla^\mu R_{\alpha\beta} + \frac{2}{3} \nabla_{(\alpha} \mathcal{R}^\mu{}_{\beta)}, \\ H_3{}_\alpha &= \frac{1}{3} \nabla_\mu \mathcal{R}^\mu{}_\alpha - \frac{1}{6} \nabla_\alpha R, \end{aligned} \quad (4.24)$$

$$\begin{aligned}
 C_4^{\mu\nu}{}_{\alpha\beta\gamma\delta} &= \frac{1}{15}R^\mu_{(\alpha|\lambda|\beta}R^\lambda{}_{\gamma}{}^\nu{}_{\delta)} + \frac{1}{20}\nabla_{(\alpha}\nabla_\beta R^\mu{}_{\gamma}{}^\nu{}_{\delta)}, \\
 E_4^\mu{}_{\alpha\beta\gamma} &= -\frac{1}{15}R^\mu{}_{\nu(\alpha|\lambda|}R^\nu{}_{\beta}{}^\lambda{}_{\gamma)} - \frac{1}{60}R^\mu{}_{(\alpha}{}^\nu{}_{\beta}R_{\gamma)\nu} - \frac{1}{4}R^\mu{}_{(\alpha}{}^\nu{}_{\beta}\mathcal{R}_{|\nu|\gamma)} \\
 &\quad + \frac{1}{10}\nabla_{(\alpha}\nabla^\mu R_{\beta\gamma)} - \frac{3}{20}\nabla_{(\alpha}\nabla_\beta R^\mu{}_{\gamma)} - \frac{1}{4}\nabla_{(\alpha}\nabla_\beta \mathcal{R}^\mu{}_{\gamma)}, \\
 G_4^\mu{}_{\alpha\beta\gamma\delta\epsilon} &= \frac{1}{40}R^\mu{}_{(\alpha|\nu|\beta}R^\nu{}_{\gamma}{}^\lambda{}_{\delta)}iF_{|\lambda|\epsilon)}, \\
 H_4{}_{\alpha\beta} &= \frac{1}{4}\mathcal{R}_{\mu(\alpha}\mathcal{R}^\mu{}_{\beta)} - \frac{1}{30}R_{\mu\alpha}R^\mu{}_{\beta} - \frac{1}{4}\nabla_{(\alpha}\nabla_{|\mu|}\mathcal{R}^\mu{}_{\beta)} + \frac{1}{60}R_{\mu\nu}R^\mu{}_{\alpha}{}^\nu{}_{\beta} \\
 &\quad + \frac{1}{60}R_{\mu\lambda\gamma\alpha}R^{\mu\lambda\gamma}{}_{\beta} + \frac{1}{40}\Delta R_{\alpha\beta} + \frac{3}{40}\nabla_\alpha\nabla_\beta R, \\
 L_4{}_{\alpha\beta\gamma\delta} &= -\frac{1}{80}R_{\mu(\alpha}{}^\nu{}_{\beta}R^\mu{}_{\gamma}iF_{|\nu|\delta)} - \frac{1}{80}R_{\mu(\alpha|\lambda|\beta}R^\lambda{}_{\gamma}{}^{\mu\nu}iF_{|\nu|\delta)} - \frac{1}{24}\mathcal{R}_{\mu(\alpha}R^\mu{}_{\beta}{}^\nu{}_{\gamma}iF_{|\nu|\delta)}, \\
 O_4{}_{\alpha\beta\gamma\delta\epsilon\kappa} &= \frac{1}{576}R_{\mu(\alpha}{}^\nu{}_{\beta}R^\mu{}_{\gamma}{}^\lambda{}_{\delta)}iF_{|\nu|\epsilon}iF_{|\lambda|\kappa)}. \tag{4.25}
 \end{aligned}$$

Here and everywhere below the parenthesis denote the complete symmetrization over all indices enclosed; the vertical lines indicate the indices excluded from the symmetrization.

4.3 Perturbation Theory for the Heat Semigroup

Now, by using the perturbative expansion (4.6) of the operator $\tilde{\mathcal{L}}$ and recalling that $\mathcal{D}^2 \sim \varepsilon^{-2}$ and $t \sim \varepsilon^2$, we see that the operator $t\mathcal{D}^2$ is of zero order and the operator $t\mathcal{L}_k$, $k \geq 2$, is of (higher) order ε^k . Therefore, we can consider the terms $t\mathcal{L}_k$ with $k \geq 2$ as a perturbation.

In order to evaluate the heat semigroup we utilize the Volterra series for the exponent of two non-commuting operators. Let X be an operator and Y be a perturbation (say of order one in a small parameter). Then

$$\exp(X + Y) = T \exp X, \tag{4.26}$$

where

$$T = I + \sum_{k=1}^{\infty} \int_0^1 d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_2} d\tau_1 \tilde{Y}(\tau_1)\tilde{Y}(\tau_2)\cdots\tilde{Y}(\tau_k) \tag{4.27}$$

and

$$\tilde{Y}(\tau) = e^{\tau X} Y e^{-\tau X}. \tag{4.28}$$

By using the above series for the operator in (4.6) we obtain

$$\exp(-t\tilde{\mathcal{L}}) = T(t) \exp(t\mathcal{D}^2), \quad (4.29)$$

where $T(t)$ is an operator defined by a formal perturbative expansion

$$T(t) \sim \sum_{k=0}^{\infty} T_k(t), \quad (4.30)$$

with $T_k(t)$ being of order ε^k . Explicitly, up to terms of fifth order we obtain

$$T_0(t) = I, \quad (4.31)$$

$$T_1(t) = 0, \quad (4.32)$$

$$T_2(t) = t \int_0^1 d\tau_1 V_2(t\tau_1), \quad (4.33)$$

$$T_3(t) = t \int_0^1 d\tau_1 V_3(t\tau_1) \quad (4.34)$$

$$T_4(t) = t \int_0^1 d\tau_1 V_4(t\tau_1) + t^2 \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 V_2(t\tau_1)V_2(t\tau_2), \quad (4.35)$$

and

$$V_k(s) = e^{s\mathcal{D}^2} \mathcal{L}_k e^{-s\mathcal{D}^2}. \quad (4.36)$$

4.4 Perturbation Theory for the Heat Kernel

As we already mentioned above the heat kernel can be computed from the heat semigroup by using the equation (3.17). By using the heat semigroup expansion from the previous section we now obtain the heat kernel in the form

$$U(t|x, x') \sim \mathcal{P}(x, x') \Delta^{1/2}(x, x') U_0(t|x, x') \sum_{k=0}^{\infty} t^{k/2} b_k(t|x, x') \quad (4.37)$$

where

$$U_0(t|x, x') = \exp(t\mathcal{D}^2) \delta(u), \quad (4.38)$$

and

$$b_k(t|x, x') = t^{-k/2} U_0^{-1}(t|x, x') T_k(t) U_0(t|x, x') . \quad (4.39)$$

Thus, the calculation of the heat kernel coefficients reduces to the evaluation of the zero-order heat kernel $U_0(t|x, x')$ and to the action of the differential operators $T_k(t)$ on it.

The zero order heat kernel $U_0(t|x, x')$ can be evaluated by using the algebraic method developed in [3, 4]. First, the heat semigroup $\exp(t\mathcal{D}^2)$ can be represented as an average over the (nilpotent) Lie group (3.10) with a Gaussian measure

$$\exp(t\mathcal{D}^2) = (4\pi t)^{-n/2} J(t) \int_{\mathbb{R}^n} dk \exp \left\{ -\frac{1}{4} k^\mu M_{\mu\nu}(t) k^\nu + k^\mu \mathcal{D}_\mu \right\} . \quad (4.40)$$

where

$$J(t) = \det \left(\frac{tiF}{\sinh(tiF)} \right)^{1/2} \quad (4.41)$$

and $M(t)$ is a symmetric matrix defined by

$$M(t) = iF \coth(tiF) . \quad (4.42)$$

We would like to stress, at this point, that here and everywhere below all the functions of the 2-form F are analytic and should be understood in terms of a power series in F .

Then by using the relation

$$\exp(k^\mu \mathcal{D}_\mu) \delta(u) = \delta(u + k) , \quad (4.43)$$

one obtains

$$U_0(t|x, x') = (4\pi t)^{-n/2} J(t) \exp \left\{ -\frac{1}{4} u^\mu M_{\mu\nu}(t) u^\nu \right\} , \quad (4.44)$$

which is nothing but the Schwinger kernel for an electromagnetic field on \mathbb{R}^n [21].

To obtain the asymptotic expansion of the heat kernel diagonal we just need to set $x = x'$ (or $u = 0$). At this point, we notice the following interesting fact. The operators $t\mathcal{L}_k$, $tV_k(t\tau)$ and $T_k(t)$ are differential operators with homogeneous polynomial coefficients (in u^μ) of order ε^k . Recall that $u \sim \varepsilon$, $t \sim \varepsilon^2$ and $F \sim \varepsilon^{-2}$, so that tF and Fuu are counted as of order zero. Since the zero order heat kernel U_0 is Gaussian, then the off-diagonal coefficients $b_k(t|x, x')$ are polynomials in u . The point we want to make now is the following.

Lemma 1. *The off-diagonal odd-order coefficients b_{2k+1} are odd order polynomials in u^μ , that is, they satisfy*

$$b_{2k+1}(t|x, x') \Big|_{u \mapsto -u} = -b_{2k+1}(t|x, x'), \quad (4.45)$$

and, therefore, vanish on the diagonal,

$$b_{2k+1}^{\text{diag}}(t) = 0. \quad (4.46)$$

Proof. We discuss the transformation properties of various quantities under the reflection of the coordinates, $u \mapsto -u$. First, we note that the operator \mathcal{D} changes sign, and, therefore, the operator $\mathcal{L}_0 = -\mathcal{D}^2$ is invariant. Next, from the general form of the operator \mathcal{L}_k discussed above we see that $\mathcal{L}_k \mapsto (-1)^k \mathcal{L}_k$. Therefore, the same is true for the operator $V_k(t\tau)$, that is, $V_k \mapsto (-1)^k V_k$.

Now, the operator $T_k(t)$ has the following general form

$$T_k = t^k \sum_{m=1}^{[k/2]} \int_0^1 d\tau_1 \cdots \int_0^{\tau_{m-1}} d\tau_m \sum_{|\mathbf{j}|=k} C_{m,\mathbf{j}} V_{j_1}(t\tau_1) \cdots V_{j_m}(t\tau_m), \quad (4.47)$$

where the summation goes over multiindex $\mathbf{j} = (j_1, \dots, j_m)$ of integers $j_1, \dots, j_m \geq 2$ such that $|\mathbf{j}| = j_1 + \cdots + j_m = k$, and $C_{m,\mathbf{j}}$ are some numerical coefficients. Therefore, the operator T_k transforms as $T_k \mapsto (-1)^k T_k$.

Since the zero-order heat kernel U_0 is invariant under the reflection of coordinates $u \mapsto -u$, we finally find that the coefficients b_k transform according to $b_k \mapsto (-1)^k b_k$. Thus, b_{2k} are even polynomials and b_{2k+1} are odd-order polynomials. □

By using this lemma and by setting $x = x'$ we obtain the asymptotic expansion of the heat kernel diagonal

$$U^{\text{diag}}(t) \sim (4\pi t)^{-n/2} J(t) \sum_{k=0}^{\infty} t^k b_{2k}^{\text{diag}}(t), \quad (4.48)$$

where the function $J(t)$ is defined in (4.41). Thus, we obtain

$$\tilde{a}_k^{\text{diag}}(t) = J(t) b_{2k}^{\text{diag}}(t). \quad (4.49)$$

4.5 Algebraic Framework

As we have shown above the evaluation of the heat semigroup is reduced to the calculation of the operators $V_k(s)$ defined by (4.36), which reduces, in turn, to the computation of general expressions

$$e^{s\mathcal{D}^2} u^{\nu_1} \dots u^{\nu_n} \mathcal{D}_{\mu_1} \dots \mathcal{D}_{\mu_m} e^{-s\mathcal{D}^2} = Z^{\nu_1}(s) \dots Z^{\nu_n}(s) A_{\mu_1}(s) \dots A_{\mu_m}(s) , \quad (4.50)$$

where

$$Z^\nu(s) = e^{s\mathcal{D}^2} u^\nu e^{-s\mathcal{D}^2} . \quad (4.51)$$

$$A_\mu(s) = e^{s\mathcal{D}^2} \mathcal{D}_\mu e^{-s\mathcal{D}^2} . \quad (4.52)$$

Obviously, the operators A_μ and Z_ν form the algebra

$$[A_\mu(s), Z^\nu(s)] = \delta_\mu^\nu, \quad [A_\mu(s), A_\nu(s)] = iF_{\mu\nu}, \quad [Z^\mu(s), Z^\nu(s)] = 0 . \quad (4.53)$$

The operators $A_\mu(s)$ and $Z^\nu(s)$ can be computed as follows. First, we notice that $A_\mu(s)$ satisfies the differential equation

$$\partial_s A_\mu(s) = \text{Ad}_{\mathcal{D}^2} A_\mu(s) , \quad (4.54)$$

with the initial condition

$$A_\mu(0) = \mathcal{D}_\mu .$$

Hereafter $\text{Ad}_{\mathcal{D}^2}$ is an operator acting as a commutator, that is,

$$\text{Ad}_{\mathcal{D}^2} A_\mu(s) \equiv [\mathcal{D}^2, A_\mu(s)] . \quad (4.55)$$

The solution of eq. (4.54) is

$$A_\mu(s) = \exp(s\text{Ad}_{\mathcal{D}^2}) \mathcal{D}_\mu , \quad (4.56)$$

which can be written in terms of series as

$$A_\mu(s) = \sum_{k=0}^{\infty} \frac{s^k}{k!} (\text{Ad}_{\mathcal{D}^2})^k \mathcal{D}_\mu . \quad (4.57)$$

Now, by using the algebra (3.10) we first obtain the commutator

$$[\mathcal{D}^2, \mathcal{D}_\mu] = -2iF_{\mu\alpha} \mathcal{D}^\alpha , \quad (4.58)$$

and then, by induction,

$$(\text{Ad}_{\mathcal{D}^2})^k \mathcal{D}_\mu = (-2i)^k F_{\mu\alpha_1} F^{\alpha_1}_{\alpha_2} \cdots F^{\alpha_{k-1}}_{\alpha_k} \mathcal{D}_{\alpha_k} = [(-2iF)^k]_{\mu\alpha} \mathcal{D}^\alpha . \quad (4.59)$$

By substituting this result in the series (4.57) we finally find that

$$A_\mu(s) = \Psi_\mu^\alpha(s) \mathcal{D}_\alpha , \quad (4.60)$$

where

$$\Psi(s) = \exp(-2siF) . \quad (4.61)$$

Similarly, for the operators $Z^\nu(s)$ we find

$$Z^\mu(s) = \exp(s\text{Ad}_{\mathcal{D}^2}) u^\mu = \sum_{k=0}^{\infty} \frac{s^k}{k!} (\text{Ad}_{\mathcal{D}^2})^k u^\mu . \quad (4.62)$$

Now, by using the commutators in (3.10), we find

$$\text{Ad}_{\mathcal{D}^2} u^\mu = [\mathcal{D}^2, u^\mu] = 2\mathcal{D}^\mu , \quad (4.63)$$

and then, by induction, we obtain, for $k \geq 2$,

$$(\text{Ad}_{\mathcal{D}^2})^k u^\mu = 2[(-2iF)^{k-1}]^{\mu\alpha} \mathcal{D}_\alpha . \quad (4.64)$$

Thus the operator $Z^\mu(s)$ in (4.52) takes the form

$$Z^\mu(s) = u^\mu - 2s\mathcal{D}^\mu + 2 \sum_{k=2}^{\infty} \frac{s^k}{k!} [(-2iF)^{k-1}]^{\mu\alpha} \mathcal{D}_\alpha . \quad (4.65)$$

This series can be easily summed up to give

$$Z^\mu(s) = u^\mu + \Omega^{\mu\alpha}(s) \mathcal{D}_\alpha , \quad (4.66)$$

where

$$\Omega(s) = \frac{1 - \exp(-2siF)}{iF} = 2 \exp(-siF) \frac{\sinh(siF)}{iF} . \quad (4.67)$$

Now, by using (4.61) and (4.67) we obtain

$$\Omega^{-1}(s) = \frac{1}{2} iF [\coth(siF) + 1] = \frac{1}{2} [M(s) + iF] . \quad (4.68)$$

We will need the symmetric and the antisymmetric parts of $\Omega^{-1}(s)$. By recalling that the matrix F is anti-symmetric it is easy to show

$$\Omega_{(\mu\nu)}^{-1}(s) = \frac{1}{2}M_{\mu\nu}(s) . \quad (4.69)$$

$$\Omega_{[\mu\nu]}^{-1}(s) = \frac{1}{2}iF_{\mu\nu} , \quad (4.70)$$

Here and everywhere below the square brackets denote the complete antisymmetrization over all indices included.

For the future reference we also notice that

$$\Omega^{-1}(s)\Omega^T(s) = \Psi^{-1}(s) = \exp(2siF) , \quad (4.71)$$

Finally, we define another function

$$\Phi(s) = \Psi(s)\Omega^{-1}(s) = \left(\Omega^{-1}(s)\right)^T = \frac{1}{2} [M(s) - iF] . \quad (4.72)$$

It is useful to remember that the functions Ψ , $F\Omega$ and $\Phi\Omega$ are dimensionless.

4.6 Flat Connection

Next, we transform the operators Z^μ to define new (time-dependent) derivative operators by

$$D_\mu(s) = \Omega_{\mu\nu}^{-1}(s)Z^\nu(s) . \quad (4.73)$$

By using the explicit form of the operators Z^μ and \mathcal{D}_μ we have

$$\begin{aligned} D_\mu(s) &= \mathcal{D}_\mu + \Omega_{\mu\rho}^{-1}(s)u^\rho \\ &= \bar{\nabla}_\mu + \frac{1}{2}M_{\mu\rho}(s)u^\rho . \end{aligned} \quad (4.74)$$

Since the operators Z^μ commute, the operators $D_\mu(s)$ obviously commute as well. In other words the connection D_μ is flat. Therefore, it can also be written as

$$D_\mu(s) = e^{-\Theta(s)}\bar{\nabla}_\mu e^{\Theta(s)} , \quad (4.75)$$

where,

$$\Theta(s) = \frac{1}{4}u^\mu M_{\mu\nu}(s)u^\nu \quad (4.76)$$

Now, we can rewrite the operators $A_\mu(s)$ and $Z^\mu(s)$ in (4.60) and (4.66) in terms of the operators $D_\mu(s)$

$$\begin{aligned} A_\mu(s) &= \Psi_\mu{}^\alpha(s) \left(D_\alpha(s) - \Omega_{\alpha\rho}^{-1}(s)u^\rho \right) , \\ Z^\mu(s) &= \Omega^{\mu\alpha}(s) D_\alpha(s) . \end{aligned} \quad (4.77)$$

5 Evaluation of the Operator T

The perturbative expansion of the operator T is given by the eq. (4.30), with the operators T_k being integrals of the operators $V_k(s)$ and their product. Thus, according to (4.33)-(4.35), to compute the operator T up to the fourth order we need to compute the operators $V_2(s)$, $V_3(s)$, $V_4(s)$ and $V_2(s_1)V_2(s_2)$.

5.1 Second Order

Now, by using the explicit expression for \mathcal{L}_2 given by eqs. (4.9), (4.16) and (4.23), utilizing the results of the Section 3, exploiting eqs. (4.77), (B.2) and (B.3), using eqs. (4.61), (4.67), (4.71) and (4.72) after some straightforward but cumbersome calculations we obtain

$$\begin{aligned} V_2(s) &= \frac{1}{6}R + N_{(2)}^\sigma D_\sigma + P_{(2)}^{\gamma\delta} D_\gamma D_\delta + W_{(2)}^{\sigma\gamma\delta} D_\sigma D_\gamma D_\delta \\ &+ Q_{(2)}^{\rho\sigma\gamma\delta} D_\rho D_\sigma D_\gamma D_\delta, \end{aligned} \quad (5.1)$$

where

$$N_{(2)}^\sigma = \left(\mathcal{R}^\mu{}_\alpha - \frac{1}{3}R^\mu{}_\alpha \right) \Omega^{\alpha\sigma} \Phi_{\mu\eta} u^\eta, \quad (5.2)$$

$$\begin{aligned} P_{(2)}^{\gamma\delta} &= \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta \Omega^{\alpha(\gamma} \Omega^{\beta)\delta} \left[\Phi_{\mu\kappa} \Phi_{\nu\sigma} u^\kappa u^\sigma - \frac{1}{2}M_{\mu\nu} \right] \\ &+ \frac{1}{24}R^\nu{}_\rho \Omega^{\rho(\gamma} \left[\delta_{\nu}{}^{\delta)} + 7\Psi_{\nu}{}^{\delta)} \right] - \mathcal{R}^\nu{}_\beta \Omega^{\beta(\gamma} \Psi_{\nu}{}^{\delta)}, \end{aligned} \quad (5.3)$$

$$W_{(2)}^{\sigma\delta\gamma} = -\frac{1}{12}R^\mu{}_\alpha{}^\nu{}_\beta \Omega^{\alpha(\sigma} \Omega^{\beta)\delta} \left[\delta_{\nu}{}^{\gamma)} + 7\Psi_{\nu}{}^{\gamma)} \right] \Phi_{\mu\kappa} u^\kappa, \quad (5.4)$$

$$Q_{(2)}^{\rho\sigma\delta\gamma} = \frac{1}{12}R^\mu{}_\alpha{}^\nu{}_\beta \Omega^{\alpha(\rho} \Omega^{\beta)\sigma} \Psi_{\mu}{}^{\delta} \left[\delta_{\nu}{}^{\gamma)} + 3\Psi_{\nu}{}^{\gamma)} \right]. \quad (5.5)$$

Note that all these coefficients as well as the operators D_μ depend on the time variable s . We will indicate explicitly the dependence of various quantities on the time parameter only in the cases when it causes confusion, in particular, when there are two time parameters.

5.2 Third Order

Similarly, by using the explicit expression for \mathcal{L}_3 given by (4.9), (4.19) and (4.24), utilizing the results of the Section 3, exploiting eqs. (4.77), (B.2) and (B.3), using

eqs. (4.61), (4.67), (4.71) and (4.72) after some straightforward but cumbersome calculations we obtain

$$V_3(s) = N_{(3)}^\sigma D_\sigma + P_{(3)}^{\sigma\rho} D_\sigma D_\rho + W_{(3)}^{\sigma\rho\iota} D_\sigma D_\rho D_\iota + Q_{(3)}^{\sigma\rho\iota\epsilon} D_\sigma D_\rho D_\iota D_\epsilon + Y_{(3)}^{\sigma\rho\iota\epsilon\kappa} D_\sigma D_\rho D_\iota D_\epsilon D_\kappa, \quad (5.6)$$

where

$$N_{(3)}^\sigma = -\frac{1}{6} \left(\nabla_\alpha R + 2 \nabla_\mu \mathcal{R}^\mu_\alpha \right) \Omega^{\alpha\sigma}, \quad (5.7)$$

$$P_{(3)}^{\gamma\delta} = -\frac{1}{6} \left(\nabla^\mu R_{\alpha\beta} - 2 \nabla_\alpha R^\mu_\beta + 4 \nabla_\alpha \mathcal{R}^\mu_\beta \right) \Omega^{\alpha(\gamma} \Omega^{|\beta|\delta)} \Phi_{\mu\kappa} u^\kappa, \quad (5.8)$$

$$W_{(3)}^{\sigma\gamma\delta} = -\frac{1}{6} \nabla_\alpha R^\mu_{\beta\ \rho} \Omega^{\alpha(\sigma} \Omega^{|\beta|\gamma} \Omega^{|\rho|\delta)} \left[\Phi_{\mu\kappa} \Phi_{\nu\epsilon} u^\kappa u^\epsilon - \frac{1}{2} M_{\mu\nu} \right] + \frac{1}{6} \left(\nabla^\mu R_{\alpha\beta} - 2 \nabla_{(\alpha} R^\mu_{\beta)} + 4 \nabla_{(\alpha} \mathcal{R}^\mu_{\beta)} \right) \Omega^{\alpha(\sigma} \Omega^{|\beta|\gamma} \Psi_\mu^{\delta)}, \quad (5.9)$$

$$Q_{(3)}^{\rho\sigma\gamma\delta} = \frac{1}{3} \nabla_\alpha R^\mu_{\beta\ \nu} \Omega^{\alpha(\rho} \Omega^{|\beta|\sigma} \Omega^{|\epsilon|\gamma} \Psi_\nu^{\delta)} \Phi_{\mu\kappa} u^\kappa, \quad (5.10)$$

$$Y_{(3)}^{\rho\sigma\gamma\delta\epsilon} = -\frac{1}{6} \nabla_{(\alpha} R^\mu_{\beta\ \eta)} \Omega^{\alpha(\rho} \Omega^{|\beta|\sigma} \Omega^{|\eta|\gamma} \Psi_\mu^{\delta)} \Psi_\nu^{\epsilon)}. \quad (5.11)$$

Here again, for simplicity, we omitted the dependence of the coefficient functions and the derivatives on the time variable s .

5.3 Fourth Order

5.3.1 Operator $V_4(s)$

By taking into account the definition of \mathcal{L}_4 in (4.9) by using eqs. (4.20)-(4.22), (4.77), (B.2) and (B.3), and the explicit form of the functions Ψ and Ω , we obtain

$$V_4(s) = P_{(4)}^{\sigma\rho} D_\sigma D_\rho + W_{(4)}^{\sigma\rho\iota} D_\sigma D_\rho D_\iota + Q_{(4)}^{\sigma\rho\iota\epsilon} D_\sigma D_\rho D_\iota D_\epsilon + Y_{(4)}^{\sigma\rho\iota\epsilon\kappa} D_\sigma D_\rho D_\iota D_\epsilon D_\kappa + S_{(4)}^{\sigma\rho\iota\epsilon\kappa\lambda} D_\sigma D_\rho D_\iota D_\epsilon D_\kappa D_\lambda, \quad (5.12)$$

where

$$\begin{aligned}
 P_{(4)}^{\sigma\rho} &= \frac{1}{60} \left[R_{\mu\nu} R^\mu{}_\alpha{}^\nu{}_\beta + R_{\mu\nu\lambda\alpha} R^{\mu\nu\lambda}{}_\beta - 2R^\mu{}_\alpha R_{\mu\beta} \right] \Omega^{\alpha(\rho} \Omega^{|\beta|\sigma)} \\
 &+ \frac{1}{40} \left[\Delta R_{\alpha\beta} + 3\nabla_\alpha \nabla_\beta R \right] \Omega^{\alpha(\rho} \Omega^{|\beta|\sigma)} \\
 &+ \frac{1}{4} \left[\mathcal{R}_{\mu\alpha} \mathcal{R}^\mu{}_\beta + \nabla_\alpha \nabla_\mu \mathcal{R}^\mu{}_\beta \right] \Omega^{\alpha(\rho} \Omega^{|\beta|\sigma)}, \tag{5.13}
 \end{aligned}$$

$$\begin{aligned}
 W_{(4)}^{\sigma\rho\iota} &= \frac{1}{60} \left[6\nabla_\alpha \nabla^\mu R_{\beta\gamma} + 15\nabla_\alpha \nabla_\beta \mathcal{R}^\mu{}_\gamma + 15R^\mu{}_\alpha{}^\nu{}_\beta \mathcal{R}_{\gamma\nu} - 9\nabla_\alpha \nabla_\beta R^\mu{}_\gamma \right. \\
 &\quad \left. - R^\mu{}_\alpha{}^\nu{}_\beta R_{\gamma\nu} - 4R^\mu{}_{\nu\alpha\lambda} R^\nu{}_\beta{}^\lambda{}_\gamma \right] \Omega^{\alpha(\sigma} \Omega^{|\beta|\rho} \Omega^{|\gamma|\iota)} \Phi_{\mu\xi} u^\xi, \tag{5.14}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{Q}_{(4)}^{\sigma\rho\iota\epsilon} &= \frac{1}{300} \left[20R^\mu{}_{\alpha\lambda\beta} R^\lambda{}_\gamma{}^\nu{}_\delta + 15\nabla_\alpha \nabla_\beta R^\mu{}_\gamma{}^\nu{}_\delta \right] \Omega^{\alpha(\sigma} \Omega^{|\beta|\rho} \Omega^{|\gamma|\iota} \Omega^{|\delta|\epsilon)} \\
 &\times \left[\Phi_{\mu\xi} \Phi_{\nu\zeta} u^\xi u^\zeta - \frac{1}{2} M_{\mu\nu} \right] \\
 &+ \frac{1}{240} R^\alpha{}_\nu R^{\mu\beta\gamma\gamma} \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \left[3\delta_\mu{}^\epsilon + \Psi_\mu{}^\epsilon \right] \\
 &+ \frac{1}{240} R_\lambda{}^\mu{}_\nu{}^\alpha R^{\lambda\beta\gamma\gamma} \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \left[3\delta_\mu{}^\epsilon + 13\Psi_\mu{}^\epsilon \right] \\
 &- \frac{1}{24} \mathcal{R}_\nu{}^\alpha R^{\mu\beta\gamma\gamma} \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \left[\delta_\mu{}^\epsilon + 5\Psi_\mu{}^\epsilon \right] \\
 &+ \frac{1}{20} \left[3\nabla^\alpha \nabla^\beta R^{\mu\gamma} - 2\nabla^\alpha \nabla^\mu R^{\beta\gamma} - 5\nabla^\alpha \nabla^\beta \mathcal{R}^{\mu\gamma} \right] \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \Psi_\mu{}^\epsilon), \tag{5.15}
 \end{aligned}$$

$$\begin{aligned}
 Y_{(4)}^{\sigma\rho\iota\epsilon\kappa} &= -\frac{1}{10} \nabla^\alpha \nabla^\beta R^{\mu\gamma\nu\delta} \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \Omega_\delta{}^\epsilon \Psi_\nu{}^\kappa) \Phi_{\mu\xi} u^\xi \\
 &- \frac{1}{120} R_\lambda{}^{\alpha\mu\beta} R^{\lambda\gamma\nu\delta} \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \Omega_\delta{}^\epsilon \left[3\delta_\nu{}^\kappa + 13\Psi_\nu{}^\kappa \right] \Phi_{\mu\xi} u^\xi, \tag{5.16}
 \end{aligned}$$

$$\begin{aligned}
 S_{(4)}^{\sigma\rho\iota\epsilon\kappa\lambda} &= \frac{1}{20} \nabla^\alpha \nabla^\beta R^{\mu\gamma\nu\delta} \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \Omega_\delta{}^\epsilon \Psi_\mu{}^\kappa \Psi_\nu{}^\lambda) \\
 &+ \frac{1}{2880} R^{\eta\alpha\mu\beta} R_\eta{}^{\gamma\nu\delta} \Omega_\alpha{}^{(\sigma} \Omega_\beta{}^\rho \Omega_\gamma{}^\iota \Omega_\delta{}^\epsilon \left[62\Psi_{(\mu}{}^\kappa \delta_{\nu)}{}^\lambda \right. \\
 &\quad \left. + 125\Psi_\mu{}^\kappa \Psi_\nu{}^\lambda + 5\delta_\mu{}^\kappa \delta_\nu{}^\lambda \right]. \tag{5.17}
 \end{aligned}$$

5.3.2 Operator $V_2(s_1)V_2(s_2)$

Next, we need to compute the product of two operators $V_2(s)$ depending on different times s_1 and s_2 by using the eq. (5.1). To simplify the notation we denote the derivatives $D_\mu(s_k)$ depending on different times s_k simply by $D_\mu^{(k)}$. To present the product $V_2(s_1)V_2(s_2)$ in the “normal” form we need to move all derivative operators $D_\mu^{(1)}$ to the right and all coordinates u^ν to the left. In order to perform this task we need the commutator of the derivative operator $D_\mu^{(1)}$ with the coefficients of the operator $V_2(s_2)$. First, by using the commutators listed in Appendix B we obtain the relevant commutators

$$\left[D_{\mu_1}^{(1)} \cdots D_{\mu_n}^{(1)}, N_{(2)}^t(s_2) \right] = n f_{(\mu_1}^t(s_2) D_{\mu_2}^{(1)} \cdots D_{\mu_n}^{(1)}, \quad (5.18)$$

$$\begin{aligned} \left[D_{\mu_1}^{(1)} \cdots D_{\mu_n}^{(1)}, P_{(2)}^\eta(s_2) \right] &= n(n-1) g_{(\mu_1 \mu_2}^\eta(s_2) D_{\mu_3}^{(1)} \cdots D_{\mu_n}^{(1)} \\ &\quad + n h_{(\mu_1}^\eta(s_2) D_{\mu_2}^{(1)} \cdots D_{\mu_n}^{(1)}, \end{aligned} \quad (5.19)$$

$$\left[D_{\mu_1}^{(1)} \cdots D_{\mu_n}^{(1)}, W_{(2)}^{\eta\kappa}(s_2) \right] = n p_{(\mu_1}^{\eta\kappa}(s_2) D_{\mu_2}^{(1)} \cdots D_{\mu_n}^{(1)}, \quad (5.20)$$

where

$$f_{\lambda}^t = \left(\mathcal{R}_{\beta}^{\mu} - \frac{1}{3} R_{\beta}^{\mu} \right) \Omega^{\beta t} \Phi_{\mu\lambda}, \quad (5.21)$$

$$\begin{aligned} g_{\lambda\kappa}^\eta &= \frac{1}{3} R_{(\alpha}^{\mu}{}_{\beta)} \Omega^{\alpha\eta} \Omega^{\beta\eta} \Phi_{\mu\lambda} \Phi_{\nu\kappa}, \\ h_{\lambda}^\eta &= \frac{2}{3} R_{(\alpha}^{\mu}{}_{\beta)} \Omega^{\alpha\eta} \Omega^{\beta\eta} \Phi_{\mu\kappa} \Phi_{\nu\lambda} u^{\kappa}, \end{aligned} \quad (5.22)$$

$$p_{\lambda}^{\eta\kappa} = -\frac{1}{12} R_{\alpha}^{\mu}{}_{\beta} \Omega^{\alpha\eta} \Omega^{\beta\eta} \left[\delta_{\nu}^{\kappa} + 7 \Psi_{\nu}^{\kappa} \right] \Phi_{\mu\lambda}. \quad (5.23)$$

Next, by using the expression for the operator $V_2(s)$ in (5.1) and the non-vanishing commutators in (5.18)-(5.19) we obtain

$$V_2(s_1)V_2(s_2) = \frac{1}{36} R^2 + \frac{1}{6} R \left[V_2(s_1) + V_2(s_2) \right] + L(s_1, s_2), \quad (5.24)$$

where

$$L(s_1, s_2) = \sum_{k=1}^4 \sum_{n=0}^4 C_{(n,k)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_k}(s_1, s_2) D_{\mu_1}^{(1)} \cdots D_{\mu_n}^{(1)} D_{\nu_1}^{(2)} \cdots D_{\nu_k}^{(2)}, \quad (5.25)$$

and

$$\begin{aligned}
 C_{(0,1)}^\rho &= N_{(2)}^\alpha(s_1) f_{\alpha}^\rho(s_2), \\
 C_{(1,1)}^{\alpha\rho} &= 2N_{(2)}^\alpha(s_1) N_{(2)}^\rho(s_2) + 2P_{(2)}^{\alpha\alpha}(s_1) f_{\alpha}^\rho(s_2), \\
 C_{(2,1)}^{\alpha\beta\rho} &= 2P_{(2)}^{\alpha\beta}(s_1) N_{(2)}^\rho(s_2) + 3W_{(2)}^{\kappa\alpha\beta}(s_1) f_{\kappa}^\rho(s_2), \\
 C_{(3,1)}^{\alpha\beta\gamma\rho} &= 2W_{(2)}^{\alpha\beta\gamma}(s_1) N_{(2)}^\rho(s_2) + 4Q_{(2)}^{\lambda\alpha\beta\gamma}(s_1) f_{\lambda}^\rho(s_2), \\
 C_{(4,1)}^{\alpha\beta\gamma\delta\rho} &= 2Q_{(2)}^{\alpha\beta\gamma\delta}(s_1) N_{(2)}^\rho(s_2),
 \end{aligned} \tag{5.26}$$

$$\begin{aligned}
 C_{(0,2)}^{\rho\sigma} &= N_{(2)}^\alpha(s_1) h^{\rho\sigma}_{\alpha}(s_2) + 2P_{(2)}^{\alpha\beta}(s_1) g^{\rho\sigma}_{\alpha\beta}(s_2), \\
 C_{(1,2)}^{\alpha\rho\sigma} &= 2N_{(2)}^\alpha(s_1) P_{(2)}^{\rho\sigma}(s_2) + 2P_{(2)}^{\alpha\beta}(s_1) h^{\rho\sigma}_{\beta}(s_2) + 6W_{(2)}^{\alpha\beta\gamma}(s_1) g^{\rho\sigma}_{\beta\gamma}(s_2), \\
 C_{(2,2)}^{\alpha\beta\rho\sigma} &= 2P_{(2)}^{\alpha\beta}(s_1) P_{(2)}^{\rho\sigma}(s_2) + 3W_{(2)}^{\alpha\beta\gamma}(s_1) h^{\rho\sigma}_{\gamma}(s_2) + 12Q_{(2)}^{\alpha\beta\gamma\delta}(s_1) g^{\rho\sigma}_{\gamma\delta}(s_2), \\
 C_{(3,2)}^{\alpha\beta\gamma\rho\sigma} &= 2W_{(2)}^{\alpha\beta\gamma}(s_1) P_{(2)}^{\rho\sigma}(s_2) + 4Q_{(2)}^{\alpha\beta\gamma\delta}(s_1) h^{\rho\sigma}_{\delta}(s_2), \\
 C_{(4,2)}^{\alpha\beta\gamma\delta\rho\sigma} &= 2Q_{(2)}^{\alpha\beta\gamma\delta}(s_1) P_{(2)}^{\rho\sigma}(s_2),
 \end{aligned} \tag{5.27}$$

$$\begin{aligned}
 C_{(0,3)}^{\rho\sigma\nu} &= N_{(2)}^\alpha(s_1) p^{\rho\sigma\nu}_{\alpha}(s_2), \\
 C_{(1,3)}^{\alpha\rho\sigma\nu} &= 2N_{(2)}^\alpha(s_1) W_{(2)}^{\rho\sigma\nu}(s_2) + 2P_{(2)}^{\mu\alpha}(s_1) p^{\rho\sigma\nu}_{\mu}(s_2), \\
 C_{(2,3)}^{\alpha\beta\rho\sigma\nu} &= 2P_{(2)}^{\alpha\beta}(s_1) W_{(2)}^{\rho\sigma\nu}(s_2) + 3W_{(2)}^{\mu\alpha\beta}(s_1) p^{\rho\sigma\nu}_{\mu}(s_2), \\
 C_{(3,3)}^{\alpha\beta\gamma\rho\sigma\nu} &= 2W_{(2)}^{\alpha\beta\gamma}(s_1) W_{(2)}^{\rho\sigma\nu}(s_2) + 4Q_{(2)}^{\mu\alpha\beta\gamma}(s_1) p^{\rho\sigma\nu}_{\mu}(s_2), \\
 C_{(4,3)}^{\alpha\beta\gamma\delta\rho\sigma\nu} &= 2Q_{(2)}^{\alpha\beta\gamma\delta}(s_1) W_{(2)}^{\rho\sigma\nu}(s_2),
 \end{aligned} \tag{5.28}$$

$$\begin{aligned}
 C_{(1,4)}^{\alpha\rho\sigma\nu\chi} &= N_{(2)}^\alpha(s_1) Q_{(2)}^{\rho\sigma\nu\chi}(s_2), \\
 C_{(2,4)}^{\alpha\beta\rho\sigma\nu\chi} &= P_{(2)}^{\alpha\beta}(s_1) Q_{(2)}^{\rho\sigma\nu\chi}(s_2), \\
 C_{(3,4)}^{\alpha\beta\gamma\rho\sigma\nu\chi} &= W_{(2)}^{\alpha\beta\gamma}(s_1) Q_{(2)}^{\rho\sigma\nu\chi}(s_2), \\
 C_{(4,4)}^{\alpha\beta\gamma\delta\rho\sigma\nu\chi} &= Q_{(2)}^{\alpha\beta\gamma\delta}(s_1) Q_{(2)}^{\rho\sigma\nu\chi}(s_2).
 \end{aligned} \tag{5.29}$$

6 Generalized Hermite Polynomials

Thus, we reduced the calculation of the asymptotic expansion of the heat kernel to the calculation of the derivatives $D_\mu(s)$ of the zero order heat kernel $U_0(t|x, x')$

given by (4.44). The needed derivatives of the zero order heat kernel can be expressed in terms of the following symmetric tensors

$$\mathcal{H}_{\mu_1 \dots \mu_n}(s) = U_0^{-1}(t|x, x') D_{\mu_1}(s) \dots D_{\mu_n}(s) U_0(t|x, x') \quad (6.1)$$

and

$$\Xi_{\nu_1 \dots \nu_m \mu_1 \dots \mu_n}(s_1, s_2) = U_0^{-1}(t|x, x') D_{\nu_1}^{(1)} \dots D_{\nu_m}^{(1)} D_{\mu_1}^{(2)} \dots D_{\mu_n}^{(2)} U_0(t|x, x') , \quad (6.2)$$

where we denoted as before $D_\mu^{(k)} = D_\mu(s_k)$.

We recall that the derivatives $D_\mu^{(1)}$ and $D_\nu^{(2)}$ do not commute! Also, U_0 is a scalar function that depends on x and x' only through the normal coordinates u^μ . The derivative operator $D_\mu(s)$ is defined by (4.74), and, when acting on a scalar function is equal to

$$\begin{aligned} D_\mu(s) &= \frac{\partial}{\partial u^\mu} + \frac{1}{2} M_{\mu\nu}(s) u^\nu \\ &= e^{-\Theta(s)} \frac{\partial}{\partial u^\mu} e^{\Theta(s)} , \end{aligned} \quad (6.3)$$

where the tensor $M_{\mu\nu}(s)$ is defined by (4.42) and the function $\Theta(s)$ is a quadratic form defined by (4.76).

Therefore, by using the explicit form of the zero order heat kernel (4.44) we see that the tensors $\mathcal{H}_{\mu_1 \dots \mu_n}(s)$ can be written in the form

$$\mathcal{H}_{\mu_1 \dots \mu_n}(s) = \exp\{\Theta(t) - \Theta(s)\} \frac{\partial}{\partial u^{\mu_1}} \dots \frac{\partial}{\partial u^{\mu_n}} \exp\{\Theta(s) - \Theta(t)\} , \quad (6.4)$$

The tensors $\mathcal{H}_{\mu_1 \dots \mu_n}(s)$ are polynomials in u^μ . They differ from the usual Hermite polynomials of several variables (see, for example, [16]) by some normalization. That is why, we call them just Hermite polynomials. The generating function for Hermite polynomials

$$\mathcal{H}(\xi, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^{\mu_1} \dots \xi^{\mu_n} \mathcal{H}_{\mu_1 \dots \mu_n}(s) \quad (6.5)$$

can be computed as follows

$$\begin{aligned} \mathcal{H}(\xi, s) &= \exp\{\Theta(t) - \Theta(s)\} \exp\left(\xi^\mu \frac{\partial}{\partial u^\mu}\right) \exp\{\Theta(s) - \Theta(t)\} , \\ &= \exp\left\{\frac{1}{2} \xi^\alpha \Lambda_{\alpha\beta}(s) [\xi^\beta + 2u^\beta]\right\} , \end{aligned} \quad (6.6)$$

where

$$\begin{aligned}\Lambda(s) &= \frac{1}{2} [M(s) - M(t)] \\ &= \frac{1}{2} \frac{iF}{\sinh(tiF)} \frac{\sinh[(t-s)iF]}{\sinh(siF)} .\end{aligned}\quad (6.7)$$

By expanding the exponent in ξ we obtain the Hermite polynomials explicitly. They can be read off from the expression

$$\xi^{\mu_1} \dots \xi^{\mu_n} \mathcal{H}_{\mu_1 \dots \mu_n}(s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2k)!}{2^k k!} \binom{n}{2k} (\xi^\alpha \Lambda_{\alpha\beta}(s) \xi^\beta)^k (\xi^\rho \Lambda_{\rho\sigma}(s) u^\sigma)^{n-2k} . \quad (6.8)$$

For convenience some low-order Hermite polynomials are given explicitly in tensorial form in Appendix A.

Similarly, the tensors $\Xi_{\nu_1 \dots \nu_m \mu_1 \dots \mu_n}(s_1, s_2)$ can be written in the form

$$\begin{aligned}\Xi_{\nu_1 \dots \nu_m \mu_1 \dots \mu_n}(s_1, s_2) &= \exp [\Theta(t) - \Theta(s_1)] \\ &\times \frac{\partial}{\partial u^{\nu_1}} \dots \frac{\partial}{\partial u^{\nu_m}} \exp [\Theta(s_1) - \Theta(s_2)] \frac{\partial}{\partial u^{\mu_1}} \dots \frac{\partial}{\partial u^{\mu_n}} \exp [\Theta(s_2) - \Theta(t)]\end{aligned}\quad (6.9)$$

They are obviously polynomial in u^μ as well. We call them Hermite polynomials of second kind. The generating function for these polynomials is defined by

$$\Xi(\xi, \eta, s_1, s_2) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \xi^{\nu_1} \dots \xi^{\nu_m} \eta^{\mu_1} \dots \eta^{\mu_n} \Xi_{\nu_1 \dots \nu_m \mu_1 \dots \mu_n}(s_1, s_2) , \quad (6.10)$$

and can be computed as follows

$$\begin{aligned}\Xi(\xi, \eta, s_1, s_2) &= \exp \{ \Theta(t) - \Theta(s_1) \} \exp \left(\xi^\mu \frac{\partial}{\partial u^\mu} \right) \exp \{ \Theta(s_1) - \Theta(s_2) \} \\ &\times \exp \left(\eta^\nu \frac{\partial}{\partial u^\nu} \right) \exp \{ \Theta(s_2) - \Theta(t) \} , \\ &= \exp \left\{ \frac{1}{2} \xi^\alpha \Lambda_{\alpha\beta}(s_1) (\xi^\beta + 2u^\beta) + \frac{1}{2} \eta^\mu \Lambda_{\mu\nu}(s_2) (\eta^\nu + 2u^\nu) + \xi^\rho \Lambda_{\rho\sigma}(s_2) \eta^\sigma \right\} ,\end{aligned}\quad (6.11)$$

Notice that

$$\Xi(\xi, \eta, s_1, s_2) = \mathcal{H}(\xi, s_1) \mathcal{H}(\eta, s_2) \exp \{ \xi^\rho \Lambda_{\rho\sigma}(s_2) \eta^\sigma \} \quad (6.12)$$

This enables one to express all Hermite polynomials of second kind $\Xi_{(n)}(s_1, s_2)$ in terms of the Hermite polynomials $\mathcal{H}_{(m)}(s_1)$, $\mathcal{H}_{(l)}(s_2)$, and the matrix $\Lambda(s_2)$. Namely, they can be read off from the expression

$$\begin{aligned} \xi^{\nu_1} \dots \xi^{\nu_m} \eta^{\mu_1} \dots \eta^{\mu_n} \Xi_{\nu_1 \dots \nu_m \mu_1 \dots \mu_n}(s_1, s_2) &= \sum_{k=0}^{\min(m,n)} k! \binom{m}{k} \binom{n}{k} \\ &\times \xi^{\nu_1} \dots \xi^{\nu_{m-k}} \mathcal{H}_{\nu_1 \dots \nu_{m-k}}(s_1) \eta^{\mu_1} \dots \eta^{\mu_{n-k}} \mathcal{H}_{\mu_1 \dots \mu_{n-k}}(s_2) \left(\xi^\rho \Lambda_{\rho\sigma}(s_2) \eta^\sigma \right)^k \end{aligned} \quad (6.13)$$

7 Off-diagonal Coefficients b_k

By using the machinery developed above, we can now write the coefficients of the asymptotic expansion of the heat kernel in terms of generalized Hermite polynomials. We define the following quantity

$$\begin{aligned} b_{2,(1)}(t|x, x') &= \int_0^1 d\tau \left[N_{(2)}^\sigma(t\tau) \mathcal{H}_\sigma(t\tau) + P_{(2)}^{\gamma\delta}(t\tau) \mathcal{H}_{\gamma\delta}(t\tau) + W_{(2)}^{\sigma\gamma\delta}(t\tau) \mathcal{H}_{\sigma\gamma\delta}(t\tau) \right. \\ &\quad \left. + Q_{(2)}^{\rho\sigma\gamma\delta}(t\tau) \mathcal{H}_{\rho\sigma\gamma\delta}(t\tau) \right]. \end{aligned} \quad (7.1)$$

Then, by referring to the formulas (5.1), (5.6), (5.12) and (5.24) and by using the following formula for multiple integrals

$$\int_a^b d\tau_n \int_a^{\tau_n} d\tau_{n-1} \dots \int_a^{\tau_2} d\tau_1 f(\tau_1) = \frac{1}{(n-1)!} \int_a^b d\tau (b-\tau)^{n-1} f(\tau), \quad (7.2)$$

we obtain

$$b_2(t|x, x') = \frac{1}{6} R + b_{2,(1)}(t|x, x'), \quad (7.3)$$

$$\begin{aligned} b_3(t|x, x') &= t^{-1/2} \int_0^1 d\tau \left[N_{(3)}^\sigma(t\tau) \mathcal{H}_\sigma(t\tau) + P_{(3)}^{\gamma\delta}(t\tau) \mathcal{H}_{\gamma\delta}(t\tau) + W_{(3)}^{\sigma\gamma\delta}(t\tau) \mathcal{H}_{\sigma\gamma\delta}(t\tau) \right. \\ &\quad \left. + Q_{(3)}^{\rho\sigma\gamma\delta}(t\tau) \mathcal{H}_{\rho\sigma\gamma\delta}(t\tau) + Y_{(3)}^{\mu\rho\sigma\gamma\delta}(t\tau) \mathcal{H}_{\mu\rho\sigma\gamma\delta}(t\tau) \right]. \end{aligned} \quad (7.4)$$

$$\begin{aligned}
 b_4(t|x, x') &= \frac{1}{72}R^2 + \frac{1}{6}Rb_{2,(1)}(t|x, x') \\
 &+ t^{-1} \int_0^1 d\tau \left[P_{(4)}^{\epsilon\epsilon}(t\tau)\mathcal{H}_{\epsilon\epsilon}(t\tau) + W_{(4)}^{\epsilon\epsilon K}(t\tau)\mathcal{H}_{\epsilon\epsilon K}(t\tau) \right. \\
 &+ Q_{(4)}^{\epsilon\epsilon K\lambda}(t\tau)\mathcal{H}_{\epsilon\epsilon K\lambda}(t\tau) + Y_{(4)}^{\epsilon\epsilon K\lambda\eta}(t\tau)\mathcal{H}_{\epsilon\epsilon K\lambda\eta}(t\tau) + S_{(4)}^{\epsilon\epsilon K\lambda\eta\gamma}(t\tau)\mathcal{H}_{\epsilon\epsilon K\lambda\eta\gamma}(t\tau) \left. \right] \\
 &+ \sum_{k=1}^4 \sum_{n=0}^4 \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 C_{(n,k)}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_k}(t\tau_1, t\tau_2) \Xi_{\mu_1 \dots \mu_n \nu_1 \dots \nu_k}(t\tau_1, t\tau_2) .
 \end{aligned} \tag{7.5}$$

8 Diagonal Coefficients b_k

In order to obtain the diagonal values $b_k^{\text{diag}}(t)$ of the coefficients $b_k(t|x, x')$ we just need to set $u = 0$ in eqs. (7.3), (7.4) and (7.5). For the rest of this section we will employ the usual convention of denoting the coincidence limit by square brackets, that is,

$$[f(u)]^{\text{diag}} = f(0). \tag{8.1}$$

By inspection of the equation defining the generalized Hermite polynomials in Appendix A one can easily notice that, in the coincidence limit, all the ones with an odd number of indices vanish identically, namely

$$[\mathcal{H}_{\mu_1 \dots \mu_{2n+1}}]^{\text{diag}} = 0. \tag{8.2}$$

By using the last remark we have the following expression for the coincidence limit of (7.3), i.e.

$$b_2^{\text{diag}}(t) = \frac{1}{6}R + b_{2,(1)}^{\text{diag}}(t), \tag{8.3}$$

where

$$b_{2,(1)}^{\text{diag}}(t) = \int_0^1 d\tau \left[P_{(2)}^{\gamma\delta}(t\tau)\mathcal{H}_{\gamma\delta}(t\tau) + Q_{(2)}^{\rho\sigma\gamma\delta}(t\tau)\mathcal{H}_{\rho\sigma\gamma\delta}(t\tau) \right]^{\text{diag}}. \tag{8.4}$$

By using the explicit form of the coefficients $P_{(2)}$, $Q_{(2)}$ and the generalized Hermite polynomials in Appendix A, we obtain

$$b_{2,(1)}^{\text{diag}}(t) = J_{(1)}^{\alpha\beta}{}_{\mu\nu}(t)R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} + J_{(2)}^{\mu\nu}(t)R_{\mu\nu} + J_{(3)}^{\mu\nu}(t)\mathcal{R}_{\mu\nu}, \tag{8.5}$$

where

$$J_{(1)}^{\alpha\beta}{}_{\mu\nu}(t) = \int_0^1 d\tau \left\{ -\frac{1}{6} \Omega^{\alpha\gamma} \Omega^{\beta\delta} M_{\mu\nu} \Lambda_{\gamma\delta} + \frac{1}{4} (\delta_\nu{}^\gamma + 3\Psi_\nu{}^\gamma) \Omega^{\alpha\rho} \Omega^{\beta\sigma} \Psi_\mu{}^\delta \Lambda_{(\rho\sigma} \Lambda_{\delta)\gamma} \right\}, \quad (8.6)$$

$$J_{(2)}^{\mu\nu}(t) = \frac{1}{24} \int_0^1 d\tau (\delta_\nu{}^\delta + 7\Psi_\nu{}^\delta) \Omega^{\mu\gamma} \Lambda_\gamma{}^\delta, \quad (8.7)$$

$$J_{(3)}^{\mu\nu}(t) = \int_0^1 d\tau \Omega^{\mu}{}_\gamma \Psi^{\nu\delta} \Lambda^\gamma{}_\delta. \quad (8.8)$$

Here all functions in the integrals depend on $t\tau$.

Next, we introduce the following matrices

$$\mathcal{A}(s) = \Omega(s) \Lambda(s) = \frac{1}{2} \frac{\exp[(t-2s)iF] - \exp(-tiF)}{\sinh(tiF)}. \quad (8.9)$$

$$\mathcal{B}(s) = \Omega(s) \Lambda(s) \Omega(s)^T = \frac{\coth(tiF)}{iF} - \frac{\cosh[(t-2s)iF]}{iF \sinh(tiF)}, \quad (8.10)$$

$$\begin{aligned} \Gamma(s) &= \Omega^{-1}(s) - \frac{1}{4} \Psi(s) \Lambda(s) - \frac{3}{4} \Lambda(s) \\ &= \frac{1}{8} \left(3iF \coth(tiF) + \frac{iF}{\sinh(tiF)} \cosh[(t-2s)iF] \right). \end{aligned} \quad (8.11)$$

Then, by using the relation

$$\Omega(s) \Lambda(s) \Psi(s)^T = \Omega^T(s) \Lambda(s) = \mathcal{A}^T(s) \quad (8.12)$$

we obtain

$$J_{(1)}^{\alpha\beta}{}_{\mu\nu}(t) = \int_0^1 d\tau \left\{ -\frac{1}{3} \mathcal{B}^{\alpha\beta}(t\tau) \Gamma_{(\mu\nu)}(t\tau) + \frac{1}{6} \left(\mathcal{A}_\mu^{(\alpha}(t\tau) \mathcal{A}^{\beta)}{}_\nu(t\tau) + 3 \mathcal{A}_{(\mu}{}^\alpha(t\tau) \mathcal{A}_{\nu)}{}^\beta(t\tau) \right) \right\}, \quad (8.13)$$

$$J_{(2)}^{\mu\nu}(t) = \frac{1}{3} \int_0^1 d\tau \mathcal{A}^{(\mu\nu)}(t\tau) = \frac{1}{6} \delta^{\mu\nu}, \quad (8.14)$$

$$J_{(3)}^{\mu\nu}(t) = - \int_0^1 d\tau \mathcal{A}^{[\mu\nu]}(t\tau) = -\frac{1}{2} \left(\frac{1}{tiF} - \coth(tiF) \right)^{[\mu\nu]}. \quad (8.15)$$

Unfortunately the integral $J_{(1)}^{\alpha\beta}{}_{\mu\nu}$ can not be computed explicitly, in general.

As we already mentioned above all odd order coefficients b_{2k+1} have zero diagonal values. We see this directly for the coefficient b_3 , which is given by (7.4). That is, by recalling the formulas in (5.7) through (5.11) and the remark (8.2) we have

$$b_3^{\text{diag}}(t) = 0. \quad (8.16)$$

Finally, we evaluate the diagonal values of fourth order coefficient b_4 given by (7.5). It can be written as follows

$$b_4^{\text{diag}}(t) = \frac{1}{72} R^2 + \frac{1}{6} R b_{2,(1)}^{\text{diag}}(t) + b_{4,(2)}^{\text{diag}}(t) + b_{4,(3)}^{\text{diag}}(t). \quad (8.17)$$

By noticing that for odd $n+k$, the diagonal values of the coefficients $C_{(n,k)}$ vanish,

$$\left[C_{(n,k)}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_k} \right]^{\text{diag}} = 0, \quad (8.18)$$

and by using the explicit form of Hermite polynomials and the generating function (6.12) we obtain

$$b_{4,(2)}^{\text{diag}}(t) = t^{-1} \int_0^1 d\tau \left\{ P_{(4)}^{\iota\epsilon}(t\tau) \Lambda_{\iota\epsilon}(t\tau) + 3 \left[Q_{(4)}^{\iota\epsilon\kappa\lambda}(t\tau) \right]^{\text{diag}} \Lambda_{\iota\epsilon}(t\tau) \Lambda_{\kappa\lambda}(t\tau) + 15 S_{(4)}^{\iota\epsilon\kappa\lambda\eta\gamma}(t\tau) \Lambda_{\iota\epsilon}(t\tau) \Lambda_{\kappa\lambda}(t\tau) \Lambda_{\eta\gamma}(t\tau) \right\}, \quad (8.19)$$

$$\begin{aligned}
 b_{4,(3)}^{\text{diag}}(t) = & \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \left\{ 2[P_{(2)}^{\alpha}(\tau_1)]^{\text{diag}} f^{\rho}_{\iota}(\tau_2) \Lambda_{\alpha\rho}^{(2)} + 2[P_{(2)}^{\alpha\beta}(\tau_1)]^{\text{diag}} g^{(\rho\sigma)}_{\alpha\beta}(\tau_2) \Lambda_{\rho\sigma}^{(2)} \right. \\
 & + 12Q_{(2)}^{\lambda\alpha\beta\gamma}(\tau_1) f^{\rho}_{\lambda}(\tau_2) \Lambda_{\alpha\beta}^{(1)} \Lambda_{\gamma\rho}^{(2)} + \left(2[P_{(2)}^{\alpha\beta}(\tau_1)]^{\text{diag}} [P_{(2)}^{\rho\sigma}(\tau_2)]^{\text{diag}} \right. \\
 & + 12Q_{(2)}^{\alpha\beta\gamma\delta}(\tau_1) g^{(\rho\sigma)}_{\gamma\delta}(\tau_2) \left(\Lambda_{\alpha\beta}^{(1)} \Lambda_{\rho\sigma}^{(2)} + 2\Lambda_{\alpha\rho}^{(2)} \Lambda_{\beta\sigma}^{(2)} \right) \\
 & + 6[P_{(2)}^{\mu\alpha}(\tau_1)]^{\text{diag}} p^{(\rho\sigma\nu)}_{\mu}(\tau_2) \Lambda_{\alpha\rho}^{(2)} \Lambda_{\sigma\nu}^{(2)} \\
 & + 2Q_{(2)}^{\alpha\beta\gamma\delta}(\tau_1) [P_{(2)}^{\rho\sigma}(\tau_2)]^{\text{diag}} \left(3\Lambda_{\alpha\beta}^{(1)} \Lambda_{\gamma\delta}^{(1)} \Lambda_{\rho\sigma}^{(2)} + 12\Lambda_{\alpha\beta}^{(1)} \Lambda_{\gamma\rho}^{(2)} \Lambda_{\delta\sigma}^{(2)} \right) \\
 & + 4Q_{(2)}^{\mu\alpha\beta\gamma}(\tau_1) p^{(\rho\sigma\nu)}_{\mu}(\tau_2) \left(9\Lambda_{\alpha\beta}^{(1)} \Lambda_{\gamma\rho}^{(2)} \Lambda_{\sigma\nu}^{(2)} + 6\Lambda_{\alpha\rho}^{(2)} \Lambda_{\beta\sigma}^{(2)} \Lambda_{\gamma\nu}^{(2)} \right) \\
 & + [P_{(2)}^{\alpha\beta}(\tau_1)]^{\text{diag}} Q_{(2)}^{\rho\sigma\nu\chi}(\tau_2) \left(3\Lambda_{\alpha\beta}^{(1)} \Lambda_{\rho\sigma}^{(2)} \Lambda_{\nu\chi}^{(2)} + 12\Lambda_{\alpha\rho}^{(2)} \Lambda_{\beta\sigma}^{(2)} \Lambda_{\nu\chi}^{(2)} \right) \\
 & + Q_{(2)}^{\alpha\beta\gamma\delta}(\tau_1) Q_{(2)}^{\rho\sigma\nu\chi}(\tau_2) \left(9\Lambda_{\alpha\beta}^{(1)} \Lambda_{\gamma\delta}^{(1)} \Lambda_{\rho\sigma}^{(2)} \Lambda_{\nu\chi}^{(2)} + 72\Lambda_{\alpha\beta}^{(1)} \Lambda_{\rho\gamma}^{(2)} \Lambda_{\sigma\delta}^{(2)} \Lambda_{\nu\chi}^{(2)} \right. \\
 & \left. \left. + 24\Lambda_{\alpha\rho}^{(2)} \Lambda_{\beta\sigma}^{(2)} \Lambda_{\gamma\nu}^{(2)} \Lambda_{\delta\chi}^{(2)} \right) \right\}, \tag{8.20}
 \end{aligned}$$

where the superscript on the matrix Λ denotes its dependence on either $t\tau_1$ or $t\tau_2$.

We see that the scalar curvature appears only in the term $b_{2,(1)}^{\text{diag}}(t)$. Now, the term $b_{4,(2)}^{\text{diag}}(t)$ only contains derivatives of the curvature and quantities which are quadratic in the curvature with some of their indices contracted. It has the following form

$$\begin{aligned}
 b_{4,(2)}^{\text{diag}}(t) = & \frac{1}{60} B_{\alpha\beta}(t) R_{\mu\nu\lambda}^{\alpha} R^{\mu\nu\lambda\beta} + A_{\lambda\alpha\gamma\beta}^{(1)}(t) R_{\mu}^{\lambda}{}_{\nu}^{\alpha} R^{\mu\gamma\nu\beta} + A_{\alpha\mu\beta\gamma\gamma\delta}^{(2)}(t) R_{\eta}^{\alpha\mu\beta} R^{\eta\gamma\nu\delta} \\
 & + \frac{1}{60} B_{\alpha\beta}(t) R_{\mu\nu} R^{\mu\alpha\nu\beta} + A_{\alpha\mu\beta\gamma}^{(3)}(t) R^{\alpha}{}_{\nu} R^{\mu\beta\nu\gamma} - \frac{1}{30} B_{\alpha\beta}(t) R_{\mu}^{\alpha} R^{\mu\beta} \\
 & + A_{\alpha\mu\beta\gamma}^{(4)}(t) \mathcal{R}_{\nu}^{\alpha} R^{\mu\beta\nu\gamma} + \frac{1}{4} B_{\alpha\beta}(t) \mathcal{R}_{\nu}^{\alpha} \mathcal{R}^{\nu\beta} \\
 & + A_{\alpha\beta\mu\gamma\gamma\delta}^{(5)}(t) \nabla^{\alpha} \nabla^{\beta} R^{\mu\gamma\nu\delta} + A_{\alpha\beta\mu\nu}^{(6)}(t) \nabla^{\alpha} \nabla^{\beta} R^{\mu\nu} + \frac{1}{40} B_{\alpha\beta}(t) \Delta R^{\alpha\beta} \\
 & + \frac{3}{40} B_{\alpha\beta}(t) \nabla^{\alpha} \nabla^{\beta} R + A_{\alpha\beta\mu\nu}^{(7)}(t) \nabla^{\alpha} \nabla^{\beta} \mathcal{R}^{\mu\nu} + \frac{1}{4} B_{\alpha\beta}(t) \nabla^{\alpha} \nabla_{\mu} \mathcal{R}^{\mu\beta}, \tag{8.21}
 \end{aligned}$$

Here the tensors $a^{(i)}(s)$ are functions that only depend on F (but not on the Rie-

mann curvature) defined by

$$a^{(1)}_{\lambda\alpha\gamma\beta}(s) = \frac{3}{80}\mathcal{B}_{(\alpha\gamma}\mathcal{A}_{\beta)\lambda} + \frac{13}{80}\mathcal{B}_{(\alpha\gamma}\mathcal{A}_{|\lambda|\beta)} , \quad (8.22)$$

$$\begin{aligned} a^{(2)}_{\alpha}{}^{\mu}{}_{\beta\gamma}{}^{\nu}{}_{\delta}(s) &= \frac{1}{480}\mathcal{B}_{\alpha(\beta}\mathcal{B}_{\gamma\delta)}\left(31(\Psi\Lambda)^{(\mu\nu)} + 65\Lambda^{\mu\nu}\right) - \frac{1}{10}M^{\mu\nu}\mathcal{B}_{\alpha(\beta}\mathcal{B}_{\gamma\delta)} \\ &+ \frac{187}{480}\mathcal{B}_{\alpha(\beta}\mathcal{A}_{\gamma}{}^{(\mu}\mathcal{A}^{\nu)}{}_{\delta)} + \frac{31}{240}\mathcal{B}_{(\beta\gamma}\mathcal{A}^{(\nu)}{}_{\delta)}\mathcal{A}_{\alpha}{}^{\mu)} \\ &+ \frac{25}{96}\mathcal{B}_{(\beta\gamma}\mathcal{A}^{(\nu)}{}_{\delta)}\mathcal{A}^{\mu)}{}_{\alpha} + \frac{1}{96}\left(\mathcal{B}_{\alpha(\beta}\mathcal{A}_{\gamma}{}^{(\mu}\mathcal{A}_{\delta)}{}^{\nu)} + \mathcal{B}_{(\beta\gamma}\mathcal{A}_{\delta)}{}^{(\nu}\mathcal{A}_{\alpha}{}^{\mu)}\right) , \end{aligned} \quad (8.23)$$

$$a^{(3)}_{\alpha\mu\beta\gamma}(s) = \frac{3}{80}\mathcal{B}_{(\alpha\beta}\mathcal{A}_{\gamma)\mu} + \frac{1}{80}\mathcal{B}_{(\alpha\beta}\mathcal{A}_{|\mu|\gamma)} , \quad (8.24)$$

$$a^{(4)}_{\alpha\mu\beta\gamma}(s) = -\frac{1}{8}\mathcal{B}_{(\alpha\beta}\mathcal{A}_{\gamma)\mu} - \frac{5}{8}\mathcal{B}_{(\alpha\beta}\mathcal{A}_{|\mu|\gamma)} , \quad (8.25)$$

$$\begin{aligned} a^{(5)}_{\alpha\beta}{}^{\mu}{}_{\gamma}{}^{\nu}{}_{\delta}(s) &= -\frac{3}{40}\mathcal{B}_{\alpha(\beta}\mathcal{B}_{\gamma\delta)}M^{\mu\nu}(t) + \frac{3}{10}\mathcal{B}_{\alpha(\beta}\mathcal{A}^{(\mu}{}_{\gamma}\mathcal{A}^{\nu)}{}_{\delta)} \\ &+ \frac{3}{10}\mathcal{B}_{(\beta\gamma}\mathcal{A}^{(\mu)}{}_{\delta)}\mathcal{A}^{\nu)}{}_{\alpha} , \end{aligned} \quad (8.26)$$

$$a^{(6)}_{\alpha\beta\mu\nu}(s) = \frac{9}{20}\mathcal{B}_{(\alpha\beta}\mathcal{A}_{|\mu|\nu)} - \frac{3}{10}\mathcal{B}_{\mu(\alpha}\mathcal{A}_{\beta\nu)} , \quad (8.27)$$

$$a^{(7)}_{\alpha\beta\mu\nu}(s) = -\frac{3}{4}\mathcal{B}_{(\alpha\beta}\mathcal{A}_{|\mu|\nu)} . \quad (8.28)$$

All functions here are evaluated at the time s (unless specified otherwise).

The term $b_{4,(3)}^{\text{diag}}(t)$ only contains quantities which are quadratic in the curvature with none of their indices contracted. It has the form

$$\begin{aligned} b_{4,(3)}^{\text{diag}}(t) &= D_{\alpha\beta\mu\nu\gamma\delta\rho\sigma}^{(1)}(t)R^{\alpha\beta\mu\nu}R^{\gamma\delta\rho\sigma} + D_{\mu\nu\alpha\beta\rho\sigma}^{(2)}(t)R^{\mu\nu}R^{\alpha\beta\rho\sigma} + D_{\mu\nu\alpha\beta}^{(3)}(t)R^{\mu\nu}R^{\alpha\beta} \\ &+ D_{\mu\nu\alpha\beta\rho\sigma}^{(4)}(t)\mathcal{R}^{\mu\nu}R^{\alpha\beta\rho\sigma} + D_{\mu\nu\alpha\beta}^{(5)}(t)\mathcal{R}^{\mu\nu}R^{\alpha\beta} + D_{\mu\nu\alpha\beta}^{(6)}(t)\mathcal{R}^{\mu\nu}\mathcal{R}^{\alpha\beta} , \end{aligned} \quad (8.29)$$

where $D_{\mu_1\cdots\mu_n}^{(i)}(t)$ are some tensor-valued functions that depend on tF . They have the form

$$D_{\mu_1\cdots\mu_n}^{(i)}(t) = \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 d_{\mu_1\cdots\mu_n}^{(i)}(t\tau_1, t\tau_2) . \quad (8.30)$$

To describe our results for the tensors $d^{(k)}$ we define new tensors

$$\mathcal{E}_{(p)\mu}{}^{\nu} = \delta_{\mu}{}^{\nu} + p \Psi_{\mu}{}^{\nu} , \quad (8.31)$$

$$\begin{aligned} \mathcal{S}_{\alpha\beta\rho\sigma\iota\kappa} &= \mathcal{B}_{\beta\sigma}\Phi_{\alpha\iota}\Phi_{\rho\kappa} - \mathcal{A}_{\beta(\iota}\mathcal{A}_{|\sigma|\kappa)}M_{\alpha\rho} \\ &- \frac{3}{4}\Omega_{\beta}{}^{(\eta}\Omega_{\sigma}{}^{\chi}\mathcal{E}_{(1)\rho}{}^{\epsilon)}\Phi_{\alpha\iota}\Lambda_{\kappa\eta}\Lambda_{\chi\epsilon} + \frac{3}{2}\Omega_{\beta}{}^{(\epsilon}\Omega_{\sigma}{}^{\lambda}\Psi_{\alpha}{}^{\eta}\mathcal{E}_{(3)\rho}{}^{\chi)}\Lambda_{\iota\epsilon}\Lambda_{\kappa\lambda}\Lambda_{\eta\chi} \end{aligned} \quad (8.32)$$

$$\begin{aligned}
\mathcal{V}_{\gamma\delta\rho\sigma\iota\kappa\eta\chi}(t\tau_1, t\tau_2) &= \Lambda_{\eta\chi}(t\tau_1) \left(\mathcal{B}_{\delta\sigma} \Phi_{\gamma\iota} \Phi_{\rho\kappa} \right)(t\tau_2) + 2 \left(\mathcal{A}_{\delta\iota} \mathcal{A}_{\sigma\kappa} \Phi_{\gamma\eta} \Phi_{\rho\chi} \right)(t\tau_2) \\
&- \frac{1}{4} \left(\Lambda_{\iota\kappa} \Lambda_{\eta\chi} \right)(t\tau_1) \left(\mathcal{B}_{\delta\sigma} M_{\gamma\rho} \right)(t\tau_2) \\
&- \Lambda_{\iota\kappa}(t\tau_1) \left(\mathcal{A}_{\delta(\chi} \mathcal{A}_{|\sigma|\eta)} M_{\gamma\rho} \right)(t\tau_2) \\
&- \frac{3}{4} \left\{ \Lambda_{\kappa\eta}(t\tau_1) \left(\Lambda_{\chi\epsilon} \Lambda_{\omega\tau} \right)(t\tau_2) + \frac{2}{3} \Lambda_{\kappa\epsilon}(t\tau_1) \left(\Lambda_{\eta\omega} \Lambda_{\chi\tau} \right)(t\tau_2) \right\} \\
&\times \left(\Omega_{\delta}^{(\epsilon} \Omega_{\sigma}^{\omega} \mathcal{E}_{(1)\rho}^{\tau)} \Phi_{\gamma\iota} \right)(t\tau_2) \\
&+ \frac{3}{16} \left\{ \left(\Lambda_{\iota\kappa} \Lambda_{\eta\chi} \right)(t\tau_1) \left(\Lambda_{\epsilon\tau} \Lambda_{\omega\lambda} \right)(t\tau_2) + 8 \Lambda_{\iota\kappa}(t\tau_1) \left(\Lambda_{\epsilon\eta} \Lambda_{\tau\chi} \Lambda_{\omega\lambda} \right)(t\tau_2) \right. \\
&+ \left. \frac{8}{3} \left(\Lambda_{\iota\epsilon} \Lambda_{\kappa\tau} \Lambda_{\eta\omega} \Lambda_{\chi\lambda} \right)(t\tau_2) \right\} \left(\Omega_{\delta}^{(\epsilon} \Omega_{\sigma}^{\tau} \Psi_{\gamma}^{\omega} \mathcal{E}_{(3)\rho}^{\lambda)} \right)(t\tau_2) . \quad (8.33)
\end{aligned}$$

Then the tensors $d^{(k)}$ have the form

$$\begin{aligned}
d_{\alpha\beta\mu\nu\gamma\delta\rho\sigma}^{(1)}(t\tau_1, t\tau_2) &= -\frac{1}{9} \left(\Omega_{\beta}^{(\iota} \Omega_{\nu}^{\kappa)} M_{\alpha\mu} \right)(t\tau_1) \left(\mathcal{B}_{\delta\sigma} \Phi_{\gamma\iota} \Phi_{\rho\kappa} \right)(t\tau_2) \\
&+ \frac{1}{9} \left(\mathcal{B}_{\beta\nu} M_{\alpha\mu} \right)(t\tau_1) \left(\mathcal{B}_{\delta\sigma} \Omega_{(\gamma\rho)}^{-1} \right)(t\tau_2) \\
&+ \frac{1}{9} \left(\Omega_{\beta}^{(\iota} \Omega_{\nu}^{\kappa)} M_{\alpha\mu} \right)(t\tau_1) \left(\mathcal{A}_{(\delta|\iota|} \mathcal{A}_{\sigma)\kappa} M_{\gamma\rho}^{-1} \right)(t\tau_2) \\
&+ \frac{1}{12} \left(\Omega_{(\beta}^{\iota} \mathcal{A}_{\nu)\eta} M_{\alpha\mu} \right)(t\tau_1) \left(\Omega_{\sigma}^{(\eta} \Omega_{\delta}^{\epsilon} \mathcal{E}_{(1)\rho}^{\chi)} \Phi_{\gamma\iota} \Lambda_{\epsilon\chi} \right)(t\tau_2) \\
&- \frac{1}{24} \left\{ \left(\mathcal{B}_{\delta\sigma} M_{\gamma\rho} \right)(t\tau_1) \left(\Lambda_{\iota\kappa} \Lambda_{\eta\chi} \right)(t\tau_2) \right. \\
&+ \left. 4 \left(\Omega_{\delta}^{(\omega} \Omega_{\sigma}^{\lambda)} M_{\gamma\rho} \right)(t\tau_1) \left(\Lambda_{\omega\iota} \Lambda_{\lambda\kappa} \Lambda_{\eta\chi} \right)(t\tau_2) \right\} \left(\Omega_{\beta}^{(\iota} \Omega_{\nu}^{\kappa} \Psi_{\alpha}^{\eta} \mathcal{E}_{(3)\mu}^{\lambda)} \right)(t\tau_2) \\
&+ \frac{1}{3} \left(\Omega_{\beta}^{(\iota} \Omega_{\nu}^{\kappa} \Psi_{\alpha}^{\eta} \mathcal{E}_{(3)\mu}^{\lambda)} \right)(t\tau_1) \mathcal{V}_{\gamma\delta\rho\sigma\iota\kappa\eta\chi}(t\tau_1, t\tau_2) , \quad (8.34)
\end{aligned}$$

$$\begin{aligned}
d_{\mu\nu\alpha\beta\rho\sigma}^{(2)}(t\tau_1, t\tau_2) &= \frac{1}{9}(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa)}M_{\alpha\rho})(t\tau_1)(\Phi_{\mu\iota}\mathcal{A}_{\nu\kappa})(t\tau_2) \\
&- \frac{1}{9}(\mathcal{B}_{\beta\sigma}M_{\alpha\rho})(t\tau_1)\mathcal{A}_{(\mu\nu)}(t\tau_2) \\
&- \frac{1}{9}\mathcal{A}_{(\mu\nu)}(t\tau_1)(\mathcal{B}_{\beta\sigma}M_{\alpha\rho})(t\tau_2) \\
&+ \frac{1}{12}(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa}\Psi_\alpha{}^\eta\mathcal{E}_{(3)\rho}{}^{\chi)}\mathcal{A}_{(\mu\nu)}\Lambda_{\iota\kappa}\Lambda_{\eta\chi})(t\tau_2) \\
&- \frac{1}{36}(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa)}M_{\alpha\rho})(t\tau_1)(\mathcal{A}_{\nu\iota}\Lambda_{\kappa\eta}\mathcal{E}_{(7)\mu}{}^\eta)(t\tau_2) \\
&+ \frac{1}{3}(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa}\Psi_\alpha{}^\eta\mathcal{E}_{(3)\rho}{}^{\chi)})(t\tau_1)\left\{-\Lambda_{\kappa\eta}(t\tau_1)(\mathcal{A}_{\nu\chi}\Phi_{\mu\iota})(t\tau_2)\right. \\
&+ \left.\frac{1}{2}(\mathcal{A}_{(\mu\nu)}\Lambda_{\iota\kappa}\Lambda_{\eta\chi})(t\tau_1) + \frac{1}{4}\Lambda_{\iota\kappa}(t\tau_1)(\mathcal{A}_{\nu(\eta}\Lambda_{\chi)\epsilon}\mathcal{E}_{(7)\mu}{}^\epsilon)(t\tau_2)\right\} \\
&+ \frac{1}{36}\Omega_\nu^{(\iota}\mathcal{E}_{(7)\mu}{}^{\kappa)}(t\tau_1)\mathcal{S}_{\alpha\beta\rho\sigma\iota\kappa}(t\tau_2) , \tag{8.35}
\end{aligned}$$

$$\begin{aligned}
d_{\mu\nu\alpha\beta}^{(3)}(t\tau_1, t\tau_2) &= -\frac{1}{36}(\Omega_\nu^{(\iota}\mathcal{E}_{(7)\mu}{}^{\kappa)})(t\tau_1)(\Phi_{\alpha\iota}\mathcal{A}_{\beta\kappa})(t\tau_2) + \frac{2}{9}\mathcal{A}_{(\mu\nu)}(t\tau_1)\mathcal{A}_{(\alpha\beta)}(t\tau_2) \\
&+ \frac{1}{144}\Omega_\nu^{(\iota}\mathcal{E}_{(7)\mu}{}^{\kappa)}(t\tau_1)(\mathcal{A}_{\beta\iota}\Lambda_{\kappa\sigma}\mathcal{E}_{(7)\alpha}{}^\sigma)(t\tau_2) . \tag{8.36}
\end{aligned}$$

$$\begin{aligned}
 d_{\mu\nu\alpha\beta\rho\sigma}^{(4)}(t\tau_1, t\tau_2) = & -\frac{1}{3}(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa)}M_{\alpha\rho})(t\tau_1)(\Phi_{\mu\iota}\mathcal{A}_{\nu\kappa})(t\tau_2) \\
 & - \frac{2}{3}(\Omega_\nu^{(\iota}\Psi_\mu^{\kappa)})(t\tau_1)(\mathcal{B}_{\beta\sigma}\Phi_{\alpha\iota}\Phi_{\rho\kappa})(t\tau_2) + \frac{1}{3}(\mathcal{B}_{\beta\sigma}M_{\alpha\rho})(t\tau_1)\mathcal{A}_{\mu\nu}(t\tau_2) \\
 & + \frac{1}{3}\mathcal{A}_{\mu\nu}(t\tau_1)(\mathcal{B}_{\beta\sigma}M_{\alpha\rho})(t\tau_2) \\
 & + \frac{2}{3}(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa)}M_{\alpha\rho})(t\tau_1)(\Psi_\mu^\eta\mathcal{A}_{\nu\iota}\Lambda_{\kappa\eta})(t\tau_2) \\
 & + \frac{2}{3}(\Omega_\nu^{(\iota}\Psi_\mu^{\kappa)})(t\tau_1)(\mathcal{A}_{\beta\iota}\mathcal{A}_{\sigma\kappa}M_{\alpha\rho})(t\tau_2) \\
 & + \frac{1}{2}(\Omega_\nu^{(\epsilon}\Psi_\mu^{\lambda)})(t\tau_1)(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa}\mathcal{E}_{(1)\rho}^\eta)\Phi_{\alpha\epsilon}\Lambda_{\lambda\iota}\Lambda_{\kappa\eta})(t\tau_2) \\
 & + \frac{1}{2}(\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa}\Psi_\alpha^\eta\mathcal{E}_{(3)\rho}^\epsilon)(t\tau_1)\{2\Lambda_{\kappa\eta}(t\tau_1)(\mathcal{A}_{\nu\epsilon}\Phi_{\mu\iota})(t\tau_2) \\
 & - (\mathcal{A}_{\mu\nu}\Lambda_{\iota\kappa}\Lambda_{\eta\epsilon})(t\tau_1) - 4\Lambda_{\iota\kappa}(t\tau_1)(\Psi_\mu^\lambda\mathcal{A}_{\nu\epsilon}\Lambda_{\eta\lambda})(t\tau_2)\} \\
 & - \frac{1}{4}\{(\mathcal{A}_{\mu\nu}\Lambda_{\iota\kappa}\Lambda_{\eta\epsilon})(t\tau_2) + 4(\Omega_\nu^{(\omega}\Psi_\mu^{\lambda)})(t\tau_1)(\Lambda_{\omega\iota}\Lambda_{\lambda\kappa}\Lambda_{\eta\epsilon})(t\tau_2)\} \\
 & \times (\Omega_\beta^{(\iota}\Omega_\sigma^{\kappa}\Psi_\alpha^\eta\mathcal{E}_{(3)\rho}^\epsilon)(t\tau_2) . \tag{8.37}
 \end{aligned}$$

$$\begin{aligned}
 d_{\mu\nu\alpha\beta}^{(5)}(t\tau_1, t\tau_2) = & \frac{1}{12}(\Omega_\beta^{(\gamma}\mathcal{E}_{(7)\alpha}^{\delta)})(t\tau_1)(\Phi_{\mu\gamma}\mathcal{A}_{\nu\delta})(t\tau_2) \\
 & + \frac{2}{3}(\Omega_\nu^{(\gamma}\Psi_\mu^{\delta)})(t\tau_1)(\Phi_{\alpha\gamma}\mathcal{A}_{\beta\delta})(t\tau_2) \\
 & - \frac{2}{3}\mathcal{A}_{(\alpha\beta)}(t\tau_1)\mathcal{A}_{\mu\nu}(t\tau_2) - \frac{2}{3}\mathcal{A}_{(\alpha\beta)}(t\tau_2)\mathcal{A}_{\mu\nu}(t\tau_1) \\
 & - \frac{1}{6}(\Omega_\beta^{(\iota}\mathcal{E}_{(7)\alpha}^{\kappa)})(t\tau_1)(\Psi_\mu^\epsilon\mathcal{A}_{\nu\kappa}\Lambda_{\iota\epsilon})(t\tau_2) \\
 & - \frac{1}{6}(\Omega_\nu^{(\iota}\Psi_\mu^{\kappa)})(t\tau_1)(C_\beta^{(\eta}\mathcal{A}_{\alpha}^{\epsilon)}\Lambda_{\iota\eta}\Lambda_{\kappa\epsilon})(t\tau_2) . \tag{8.38}
 \end{aligned}$$

$$\begin{aligned}
 d_{\mu\nu\alpha\beta}^{(6)}(t\tau_1, t\tau_2) = & -2(\Omega_\nu^{(\gamma}\Psi_\mu^{\delta)})(t\tau_1)(\Phi_{\alpha\gamma}\mathcal{A}_{\beta\delta})(t\tau_2) + 2\mathcal{A}_{\mu\nu}(t\tau_1)\mathcal{A}_{\alpha\beta}(t\tau_2) \\
 & + 4(\Omega_\nu^{(\gamma}\Psi_\mu^{\delta)})(t\tau_1)(\Psi_\alpha^\sigma\mathcal{A}_{\beta\delta}\Lambda_{\gamma\sigma})(t\tau_2) . \tag{8.39}
 \end{aligned}$$

9 Conclusions

In this paper we studied the heat kernel expansion for a Laplace operator acting on sections of a complex vector bundle over a smooth compact Riemannian man-

ifold without boundary. We assumed that the curvature F of the $U(1)$ part of the total connection (the electromagnetic field) is covariantly constant and large, so that $tF \sim 1$, that is, F is of order t^{-1} . In this situation the standard asymptotic expansion of the heat kernel as $t \rightarrow 0$ does not apply since the electromagnetic field can not be treated as a perturbation.

In order to calculate the heat kernel asymptotic expansion we use an algebraic approach in which the nilpotent algebra of the operators \mathcal{D}_μ plays a major role. In this approach the calculation of the asymptotic expansion of the heat kernel is reduced to the calculation of the asymptotic expansion of the heat semigroup and, then, to the action of differential operators on the zero-order heat kernel. Since the zero-order heat kernel has the Gaussian form the heat kernel asymptotics are expressed in terms of generalized Hermite polynomials.

The main result of this work is establishing the existence of a new non-perturbative asymptotic expansion of the heat kernel and the explicit calculation of the first three coefficients of this expansion (both off-diagonal and the diagonal ones). As far as we know, such an asymptotic expansion and the explicit form of these modified heat kernel coefficients are new.

We presented our result as explicitly as possible. Unfortunately, some of the integrals of the tensor-valued functions cannot be evaluated explicitly in full generality. They can be evaluated, in principle, by using the spectral decomposition of the two-form F ,

$$F = \sum_{k=1}^{[n/2]} B_k E_k, \quad F^2 = - \sum_{k=1}^{[n/2]} B_k^2 \Pi_k, \quad (9.1)$$

where B_k are the eigenvalues, E_k are the (2-dimensional) eigen-two-forms, and $\Pi_k = -E_k^2$ are the corresponding eigen-projections onto 2-dimensional eigenspaces. Then for any analytic function of tiF we have

$$f(tiF) = \sum_{k=1}^{[n/2]} f(tB_k) \frac{1}{2} (\Pi_k + iE_k) + \sum_{k=1}^{[n/2]} f(-tB_k) \frac{1}{2} (\Pi_k - iE_k). \quad (9.2)$$

However, this seems impractical in general case in n dimensions. It would simplify substantially in the following cases: i) there is only one eigenvalue (one magnetic field) in a corresponding two-dimensional subspace, that is, $F = B_1 E_1$ (which is essentially 2-dimensional), and ii) all eigenvalues are equal so that $F^2 = -I$ (which is only possible in even dimensions). We plan to study this problem in a future work.

The work carried on in this paper can find useful applications in various fields of theoretical physics and mathematics. For instance, our results can be applied to the study of the heat kernel asymptotic expansion on Kähler manifolds. The complex structure on Kähler manifolds is a parallel antisymmetric two-tensor which plays the role of the covariantly constant electromagnetic field. This subject is also interesting, in particular, in connection with String Theory.

Appendix A. Hermite Polynomials

The Hermite polynomials are defined by

$$\begin{aligned} \mathcal{H}_{\mu_1 \dots \mu_n} &= \exp \left\{ -\frac{1}{2} u^\alpha \Lambda_{\alpha\beta} u^\beta \right\} \frac{\partial}{\partial u^{\mu_1}} \cdots \frac{\partial}{\partial u^{\mu_n}} \exp \left\{ \frac{1}{2} u^\alpha \Lambda_{\alpha\beta} u^\beta \right\} \\ &= \left(\frac{\partial}{\partial u^{\mu_1}} + \Lambda_{\mu_1 \nu_1} u^{\nu_1} \right) \cdots \left(\frac{\partial}{\partial u^{\mu_n}} + \Lambda_{\mu_n \nu_n} u^{\nu_n} \right) \cdot 1 . \end{aligned} \quad (\text{A.1})$$

They can be computed explicitly as follows. First, let

$$\mathcal{H}_{(n)}(\xi) = \xi^{\mu_1} \cdots \xi^{\mu_n} \mathcal{H}_{\mu_1 \dots \mu_n} \quad (\text{A.2})$$

and

$$B = \xi^\mu \frac{\partial}{\partial u^\mu} , \quad A = \xi^\mu \Lambda_{\mu\nu} u^\nu . \quad (\text{A.3})$$

Then

$$\mathcal{H}_{(n)}(\xi) = (A + B)^n \cdot 1 . \quad (\text{A.4})$$

Finally, let

$$C = [B, A] = \xi^\mu \Lambda_{\mu\nu} \xi^\nu . \quad (\text{A.5})$$

Obviously, the operators A, B, C form the Heisenberg algebra

$$[B, A] = C , \quad [A, C] = [B, C] = 0 .$$

Lemma 2. *There holds,*

$$(A + B)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2k} \frac{(2k)!}{2^k k!} \binom{n}{2k} \binom{n-2k}{m} C^k A^{n-2k-m} B^m . \quad (\text{A.6})$$

Proof. Notice that $e^{t(A+B)}$ is the generating functional for $(A+B)^n$. Now, by using the Baker-Hausdorff-Campbell formula

$$e^{t(A+B)} = e^{\frac{t}{2}C} e^{tA} e^{tB} ,$$

expanding both sides in t and computing the Taylor coefficients of the right hand side we obtain the eq. (A.6). \square

By using this result we obtain an explicit expression for (A.4)

$$\mathcal{H}_{(n)}(\xi) = \xi^{\mu_1} \dots \xi^{\mu_n} \mathcal{H}_{\mu_1 \dots \mu_n} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^k k! (n-2k)!} C^k A^{n-2k} . \quad (\text{A.7})$$

By setting $A = 0$ we immediately obtain the (diagonal) values of Hermite polynomials at $u = 0$

$$\left[\mathcal{H}_{\mu_1 \dots \mu_{2n+1}} \right]^{\text{diag}} = 0 , \quad (\text{A.8})$$

$$\left[\mathcal{H}_{\mu_1 \dots \mu_{2n}} \right]^{\text{diag}} = \frac{(2n)!}{2^n n!} \Lambda_{(\mu_1 \mu_2} \dots \Lambda_{\mu_{2n-1} \mu_{2n})} . \quad (\text{A.9})$$

We list below a few low order Hermite polynomials needed for our calculation

$$\mathcal{H}_{(0)} = 1 , \quad (\text{A.10})$$

$$\mathcal{H}_{\mu_1} = \Lambda_{\mu_1 \alpha} u^\alpha , \quad (\text{A.11})$$

$$\mathcal{H}_{\mu_1 \mu_2} = \Lambda_{(\mu_1 \mu_2)} + \Lambda_{\mu_1 \alpha} \Lambda_{\mu_2 \beta} u^\alpha u^\beta , \quad (\text{A.12})$$

$$\mathcal{H}_{\mu_1 \mu_2 \mu_3} = 3 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3) \alpha} u^\alpha + \Lambda_{\mu_1 \alpha} \Lambda_{\mu_2 \beta} \Lambda_{\mu_3 \gamma} u^\alpha u^\beta u^\gamma , \quad (\text{A.13})$$

$$\begin{aligned} \mathcal{H}_{\mu_1 \mu_2 \mu_3 \mu_4} &= 3 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3 \mu_4)} + 3 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3 | \alpha |} \Lambda_{\mu_4) \beta} u^\alpha u^\beta \\ &+ \Lambda_{\mu_1 \alpha} \Lambda_{\mu_2 \beta} \Lambda_{\mu_3 \gamma} \Lambda_{\mu_4 \delta} u^\alpha u^\beta u^\gamma u^\delta . \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} \mathcal{H}_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} &= 15 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3 \mu_4} \Lambda_{\mu_5) \alpha} u^\alpha + 5 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3 | \alpha |} \Lambda_{\mu_4 | \beta |} \Lambda_{\mu_5) \gamma} u^\alpha u^\beta u^\gamma \\ &+ \Lambda_{\mu_1 \alpha} \Lambda_{\mu_2 \beta} \Lambda_{\mu_3 \gamma} \Lambda_{\mu_4 \delta} \Lambda_{\mu_5 \eta} u^\alpha u^\beta u^\gamma u^\delta u^\eta , \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} \mathcal{H}_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} &= 15 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3 \mu_4} \Lambda_{\mu_5 \mu_6)} + 45 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3 \mu_4} \Lambda_{\mu_5 | \alpha |} \Lambda_{\mu_6) \beta} u^\alpha u^\beta \\ &+ 15 \Lambda_{(\mu_1 \mu_2} \Lambda_{\mu_3 | \alpha |} \Lambda_{\mu_4 | \beta |} \Lambda_{\mu_5 | \gamma |} \Lambda_{\mu_6) \delta} u^\alpha u^\beta u^\gamma u^\delta \\ &+ \Lambda_{(\mu_1 | \alpha |} \Lambda_{\mu_2 | \beta |} \Lambda_{\mu_3 | \gamma |} \Lambda_{\mu_4 | \delta |} \Lambda_{\mu_5 | \eta |} \Lambda_{\mu_6) \epsilon} u^\alpha u^\beta u^\gamma u^\delta u^\eta u^\epsilon , \end{aligned} \quad (\text{A.16})$$

We list below some of the generalized Hermite polynomials of second kind. Now we have two sets of Hermite polynomials that depend on the quadratic forms Λ at two different times, s_1 and s_2 . Let us define

$$\mathcal{H}_{(n)}(s_1) = \xi^{\mu_1} \cdots \xi^{\mu_n} \mathcal{H}_{\mu_1 \cdots \mu_n}(s_1) , \quad (\text{A.17})$$

$$\mathcal{H}_{(n)}(s_2) = \eta^{\mu_1} \cdots \eta^{\mu_n} \mathcal{H}_{\mu_1 \cdots \mu_n}(s_2) , \quad (\text{A.18})$$

and

$$\Lambda(s_2) = \xi^\alpha \Lambda_{\alpha\beta}(s_2) \eta^\beta . \quad (\text{A.19})$$

Then from eq. (6.13) we obtain the quantities $\Xi_{(m,n)}$ that we need in our calculations

$$\Xi_{(0,1)}(s_1, s_2) = \mathcal{H}_{(1)}(s_2) \quad (\text{A.20})$$

$$\Xi_{(1,1)}(s_1, s_2) = \Lambda(s_2) + \mathcal{H}_{(1)}(s_1) \mathcal{H}_{(1)}(s_2) , \quad (\text{A.21})$$

$$\Xi_{(2,1)}(s_1, s_2) = 2\Lambda(s_2) \mathcal{H}_{(1)}(s_1) + \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(2)}(s_1) , \quad (\text{A.22})$$

$$\Xi_{(3,1)}(s_1, s_2) = 3\Lambda(s_2) \mathcal{H}_{(2)}(s_1) + \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(3)}(s_1) , \quad (\text{A.23})$$

$$\Xi_{(4,1)}(s_1, s_2) = 4\Lambda(s_2) \mathcal{H}_{(3)}(s_1) + \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(4)}(s_1) . \quad (\text{A.24})$$

$$\Xi_{(0,2)}(s_1, s_2) = \mathcal{H}_{(2)}(s_2) \quad (\text{A.25})$$

$$\Xi_{(1,2)}(s_1, s_2) = 2\Lambda(s_2) \mathcal{H}_{(1)}(s_2) + \mathcal{H}_{(2)}(s_2) \mathcal{H}_{(1)}(s_1) , \quad (\text{A.26})$$

$$\Xi_{(2,2)}(s_1, s_2) = 2\Lambda^2(s_2) + 4\Lambda(s_2) \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(1)}(s_1) + \mathcal{H}_{(2)}(s_2) \mathcal{H}_{(2)}(s_1) \quad (\text{A.27})$$

$$\begin{aligned} \Xi_{(3,2)}(s_1, s_2) &= 6\Lambda^2(s_2) \mathcal{H}_{(1)}(s_1) + 6\Lambda(s_2) \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(2)}(s_1) \\ &\quad + \mathcal{H}_{(2)}(s_2) \mathcal{H}_{(3)}(s_1) , \end{aligned} \quad (\text{A.28})$$

$$\begin{aligned} \Xi_{(4,2)}(s_1, s_2) &= 12\Lambda^2(s_2) \mathcal{H}_{(2)}(s_1) + 8\Lambda(s_2) \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(3)}(s_1) \\ &\quad + \mathcal{H}_{(2)}(s_2) \mathcal{H}_{(4)}(s_1) . \end{aligned} \quad (\text{A.29})$$

$$\Xi_{(0,3)}(s_1, s_2) = \mathcal{H}_{(3)}(s_2) \quad (\text{A.30})$$

$$\Xi_{(1,3)}(s_1, s_2) = 3\Lambda(s_2) \mathcal{H}_{(2)}(s_2) + \mathcal{H}_{(3)}(s_2) \mathcal{H}_{(1)}(s_1) , \quad (\text{A.31})$$

$$\begin{aligned} \Xi_{(2,3)}(s_1, s_2) &= 6\Lambda^2(s_2) \mathcal{H}_{(1)}(s_2) + 6\Lambda(s_2) \mathcal{H}_{(2)}(s_2) \mathcal{H}_{(1)}(s_1) \\ &\quad + \mathcal{H}_{(3)}(s_2) \mathcal{H}_{(2)}(s_1) , \end{aligned} \quad (\text{A.32})$$

$$\begin{aligned} \Xi_{(3,3)}(s_1, s_2) &= 6\Lambda^3(s_2) + 18\Lambda^2(s_2) \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(1)}(s_1) \\ &\quad + 9\Lambda(s_2) \mathcal{H}_{(2)}(s_2) \mathcal{H}_{(2)}(s_1) + \mathcal{H}_{(3)}(s_2) \mathcal{H}_{(3)}(s_1) , \end{aligned} \quad (\text{A.33})$$

$$\begin{aligned} \Xi_{(4,3)}(s_1, s_2) &= 24\Lambda^3(s_2) \mathcal{H}_{(1)}(s_1) + 36\Lambda^2(s_2) \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(2)}(s_1) \\ &\quad + 12\Lambda(s_2) \mathcal{H}_{(2)}(s_2) \mathcal{H}_{(3)}(s_1) + \mathcal{H}_{(3)}(s_2) \mathcal{H}_{(4)}(s_1) , \end{aligned} \quad (\text{A.34})$$

$$\Xi_{(0,4)}(s_1, s_2) = \mathcal{H}_{(4)}(s_2) \quad (\text{A.35})$$

$$\Xi_{(1,4)}(s_1, s_2) = 4\Lambda(s_2)\mathcal{H}_{(3)}(s_2) + \mathcal{H}_{(4)}(s_2)\mathcal{H}_{(1)}(s_1), \quad (\text{A.36})$$

$$\begin{aligned} \Xi_{(2,4)}(s_1, s_2) &= 12\Lambda^2(s_2)\mathcal{H}_{(2)}(s_2) + 8\Lambda(s_2)\mathcal{H}_{(3)}(s_2)\mathcal{H}_{(1)}(s_1) \\ &+ \mathcal{H}_{(4)}(s_2)\mathcal{H}_{(2)}(s_1), \end{aligned} \quad (\text{A.37})$$

$$\begin{aligned} \Xi_{(3,4)}(s_1, s_2) &= 24\Lambda^3(s_2)\mathcal{H}_{(1)}(s_2) + 36\Lambda^2(s_2)\mathcal{H}_{(2)}(s_2)\mathcal{H}_{(1)}(s_1) \\ &+ 12\Lambda(s_2)\mathcal{H}_{(3)}(s_2)\mathcal{H}_{(2)}(s_1) + \mathcal{H}_{(4)}(s_2)\mathcal{H}_{(3)}(s_1), \end{aligned} \quad (\text{A.38})$$

$$\begin{aligned} \Xi_{(4,4)}(s_1, s_2) &= 24\Lambda^4(s_2) + 96\Lambda^3(s_2)\mathcal{H}_{(1)}(s_2)\mathcal{H}_{(1)}(s_1) + 72\Lambda^2(s_2)\mathcal{H}_{(2)}(s_2)\mathcal{H}_{(2)}(s_1) \\ &+ 16\Lambda(s_2)\mathcal{H}_{(3)}(s_2)\mathcal{H}_{(3)}(s_1) + \mathcal{H}_{(4)}(s_2)\mathcal{H}_{(4)}(s_1), \end{aligned} \quad (\text{A.39})$$

The coincidence limit of the quantities $\Xi_{(m,n)}$, with $m+n$ odd, vanishes identically

$$[\Xi_{(m,n)}(s_1, s_2)]^{\text{diag}} = 0, \quad \text{if } (m+n) \text{ is odd} \quad (\text{A.40})$$

By recalling the coincidence limits of the Hermite polynomials we obtain the following

$$[\Xi_{(1,1)}(s_1, s_2)]^{\text{diag}} = \Lambda(s_2), \quad (\text{A.41})$$

$$[\Xi_{(3,1)}(s_1, s_2)]^{\text{diag}} = 3\Lambda(s_1)\Lambda(s_2), \quad (\text{A.42})$$

$$[\Xi_{(0,2)}(s_1, s_2)]^{\text{diag}} = \Lambda(s_2), \quad (\text{A.43})$$

$$[\Xi_{(2,2)}(s_1, s_2)]^{\text{diag}} = \Lambda(s_1)\Lambda(s_2) + 2\Lambda^2(s_2), \quad (\text{A.44})$$

$$[\Xi_{(4,2)}(s_1, s_2)]^{\text{diag}} = 3\Lambda^2(s_1)\Lambda(s_2) + 12\Lambda(s_1)\Lambda^2(s_2), \quad (\text{A.45})$$

$$[\Xi_{(1,3)}(s_1, s_2)]^{\text{diag}} = 3\Lambda^2(s_2), \quad (\text{A.46})$$

$$[\Xi_{(3,3)}(s_1, s_2)]^{\text{diag}} = 9\Lambda(s_1)\Lambda^2(s_2) + 6\Lambda^3(s_2), \quad (\text{A.47})$$

$$[\Xi_{(2,4)}(s_1, s_2)]^{\text{diag}} = 3\Lambda(s_1)\Lambda^2(s_2) + 12\Lambda^3(s_2), \quad (\text{A.48})$$

$$[\Xi_{(4,4)}(s_1, s_2)]^{\text{diag}} = 9\Lambda^2(s_1)\Lambda^2(s_2) + 72\Lambda(s_1)\Lambda^3(s_2) + 24\Lambda^4(s_2). \quad (\text{A.49})$$

Appendix B. Commutators

Lemma 3. Let D_μ and u^ν be operators satisfying the algebra

$$[D_\mu, u^\nu] = \delta_\mu^\nu, \quad [D_\mu, D_\nu] = [u^\mu, u^\nu] = 0. \quad (\text{B.1})$$

Then

$$[D_{\mu_1} \cdots D_{\mu_n}, u^\rho] = n \delta_{(\mu_1}^\rho D_{\mu_2} \cdots D_{\mu_n)} \quad (\text{B.2})$$

$$\begin{aligned} [D_{\mu_1} \cdots D_{\mu_n}, u^\rho u^\sigma] &= n(n-1) \delta_{(\mu_1}^\rho \delta_{\mu_2}^\sigma D_{\mu_3} \cdots D_{\mu_n)} \\ &+ 2n u^{(\rho} \delta^{\sigma)}_{(\mu_1} D_{\mu_2} \cdots D_{\mu_n)}. \end{aligned} \quad (\text{B.3})$$

Proof. Let $\mathcal{X}(\xi) = \xi^\mu D_\mu$ and

$$\varphi^\rho(t) = \left[e^{t\mathcal{X}(\xi)}, u^\rho \right] = \left(e^{t\mathcal{X}(\xi)} u^\rho e^{-t\mathcal{X}(\xi)} - u^\rho \right) e^{t\mathcal{X}(\xi)}, \quad (\text{B.4})$$

Then

$$e^{t\mathcal{X}(\xi)} u^\rho e^{-t\mathcal{X}(\xi)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\text{Ad}_{\mathcal{X}(\xi)} \right)^k u^\rho. \quad (\text{B.5})$$

By using the commutation relation in (B.1) we have

$$[\mathcal{X}(\xi), u^\rho] = \xi^\rho \quad (\text{B.6})$$

and, therefore,

$$e^{t\mathcal{X}(\xi)} u^\rho e^{-t\mathcal{X}(\xi)} = u^\rho + t\xi^\rho. \quad (\text{B.7})$$

Thus

$$\varphi^\rho(t) = t\xi^\rho e^{t\mathcal{X}(\xi)}. \quad (\text{B.8})$$

By expanding in Taylor series both sides of the last equation we obtain

$$\sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!} \xi^{\mu_1} \dots \xi^{\mu_{k+1}} \left[D_{(\mu_1} \dots D_{\mu_{k+1})}, u^\rho \right] = \sum_{k=0}^{\infty} \frac{t^{k+1}}{k!} \xi^{\mu_1} \dots \xi^{\mu_{k+1}} \delta_{(\mu_1}^\rho D_{\mu_2} \dots D_{\mu_{k+1})}. \quad (\text{B.9})$$

Now by equating the same powers of t in both series we obtain the claim (B.2).

The second relation can be proved in a similar manner. We introduce, in this case, the following generating function

$$\varphi^{\rho\sigma}(t) = \left[e^{t\mathcal{X}(\xi)}, u^\rho u^\sigma \right]. \quad (\text{B.10})$$

By the same argument used in the proof of the first relation we obtain that

$$\varphi^{\rho\sigma}(t) = \left[e^{t\mathcal{X}(\xi)}, u^\rho u^\sigma \right] = 2t\xi^{(\rho} u^{\sigma)} e^{t\mathcal{X}(\xi)} + t^2 \xi^\rho \xi^\sigma. \quad (\text{B.11})$$

Now, as before, by expanding the last equation in Taylor series and equating the same powers of t we obtain the claim (B.3). \square

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